BST 675 — Fall 2010 — Dr. Charnigo

In-Class Assessment

This assessment is a strictly individual activity. Textbooks, notes, calculators, computers, and technology with Internet access are prohibited. Record that which you want graded in the blue book.

[20] 1. Students commonly confuse the concepts of "mutually exclusive" and "independence". This exercise provides clarification of these two important concepts.

[10] a. For two events A_1 and A_2, state what is meant when we say that they are mutually exclusive and what is meant when we say that they are independent.

[10] b. If $P(A_1) \neq 0$ and $P(A_2) \neq 0$, show that A_1 and A_2 cannot be both independent and mutually exclusive. Hence, "independent" and "mutually exclusive" are not synonyms.

[40] 2. This exercise explores what can be said when the "empirical rule" of 95% probability within two standard deviations of the mean is inapplicable due to non-normality. We do assume, however, that X has finite mean $\mu \in \mathbb{R}$ and variance $\sigma^2 \in (0, \infty)$.

[10] a. Let $g_1(x)$, $g_2(x)$, and $g_3(x)$ be real-valued functions such that $g_1(x) \leq g_2(x) \leq g_3(x)$. State the monotonicity property of expected value.

[10] b. Put $g_1(x) := 1_{\{|x-\mu| \geq 2\sigma\}}$, $g_2(x) := (x-\mu)^2/(4\sigma^2) \times 1_{\{|x-\mu| \geq 2\sigma\}}$, and $g_3(x) := (x-\mu)^2/(4\sigma^2)$. Show that $g_1(x) \leq g_2(x) \leq g_3(x)$.

Hint. Consider two cases: (i) x for which the condition in the indicator is not satisfied; and, (ii) x for which the condition in the indicator is satisfied.

[10] c. Noting that $P(|X-\mu| \geq 2\sigma) = E[g_1(X)]$, use parts a and b to conclude that $P(|X-\mu| \geq 2\sigma) \leq (1/4)$. Thus, without normality (but with finite mean and variance), the empirical rule changes from 95% probability within two standard deviations of the mean to at least 75% probability within two standard deviations of the mean.

Remark. This is a special case of what statisticians call Chebychev's Inequality, which in its general form states that $P(|X-\mu| \geq \epsilon\sigma) \leq (1/\epsilon^2)$ for any $\epsilon > 0$.

[10] d. Let X have the exponential distribution with mean (and standard deviation) 1. Calculate $P(|X-\mu| \geq 2\sigma)$. Observe that the answer is much less than (1/4), so (1/4) is really a worst case scenario for part c rather than what we would typically anticipate.

Remark. Statisticians thus say that Chebychev's Inequality is "conservative".

[40] 3. This exercise explores computations of expected values. We assume that X has the Poisson distribution with mean $\lambda \in (0, \infty)$.

[10] a. Report the probability mass function of X and state (without proof) how $Var[X]$ is related to $E[X]$.

[10] b. Since $E[X] = \lambda$, one might guess that $E[\exp(-X)] = \exp(-\lambda)$. However, such a guess would not be correct. To see why not, let Y have the Poisson distribution with mean $\lambda \exp[-1]$. Adopting the usual notation for probability mass functions, show that $\exp[-x] f_X(x) = c(\lambda) f_Y(x)$ for nonnegative integers x, where $c(\lambda)$ is a function of λ (but not x) that you will specify.

[10] c. Continuing from part b, now apply the kernel method to calculate $E[\exp(-X)]$, confirming that $E[\exp(-X)] \neq \exp(-\lambda)$.

[10] d. Provide another example of a function $g(x)$, besides $\exp(-x)$ or a constant multiple thereof, for which $E[g(X)] \neq g(E[X])$.

Hint. Is there a choice of $g(x)$ for which $E[g(X)] - g(E[X])$ has a familiar interpretation?