BST 675 — Fall 2010 — Dr. Charnigo

Final Examination

This non-collaborative take-home final examination is due at 5:30 p.m. on Monday 13 December. By non-collaborative I mean that you are not permitted to discuss the examination with anyone other than me, until after the deadline for submission. The examination is to be submitted in hard copy, to me in person or under my office door (CPH 203-B).

[50] 1. Suppose that X has the chi-square distribution on k (positive integer) df and that Y, independent of X, has the chi-square distribution on 1 df.

[20] a. Put $U := X + Y$ and $V := X$. Find the joint probability density function of U and V. Are U and V independent? Did you expect them to be?

[15] b. Find the marginal probability density function of U by integrating the joint probability density function of U and V in dv.

Hint. You may quote without proof that, by trigonometric substitution $\sin^2 \theta := v/u$,

$$\int_0^u v^{k/2-1}/\sqrt{u-v}\ dv = 2u^{(k-1)/2} \int_0^{\pi/2} \sin^{k-1}(\theta)\ d\theta = u^{(k-1)/2}\Gamma[k/2]\Gamma[1/2]/\Gamma[(k+1)/2].$$

[15] c. As a check of your answer to part b, use moment generating functions to find the distribution of $X + Y$. You may quote without proof the well-known result that a gamma random variable with shape $\alpha(> 0)$ and scale $\beta(> 0)$ has moment generating function $(1 - \beta t)^{-\alpha}$ for $t < 1/\beta$.

[50] 2. Let X have probability density function $f(x) := 1_{\{0<x<1\}}$.

[15] a. Find the probability density function of $Y := -c^{-1} \log X$, where c is a positive constant.

[15] b. Use your answer to part a along with the kernel method to find the expected value of $Y^a \exp(-bY)$, where a and b are positive constants.

[20] c. Find the probability density function of $W := z_X$, where z_x denotes the x quantile of the standard normal distribution for $0 < x < 1$.

Hint. Letting Φ denote the cumulative distribution function of the standard normal distribution, we have $\Phi(z_x) = x$. As such, we have $g^{-1}(y) = \Phi(y)$ when $g(x) := z_x$.

Remark. Since this item can be handled similarly if z_X is replaced by $t_{k,X}$ or $\chi^2_{k,X}$, etc., the practical implication of this item is that you can simulate an observation from any continuous distribution of practical interest by first simulating an observation from the uniform distribution on $(0, 1)$ and then applying an appropriate transformation.