BST 675 — Fall 2010 — Dr. Charnigo

Solutions to Written Assignment 3

1a. Put $f_X(x) := Cx^2(1-x)1_{\{0<x<1\}}$. For $f_X(x)$ to be a probability density function, two conditions must be satisfied. First, $f_X(x)$ must be nonnegative, which imposes the constraint that $C \geq 0$. Second, $f_X(x)$ must integrate to one, which imposes the further constraint that $C = 12$ since

$$1 = \int_0^1 Cx^2(1-x)\ dx = C\int_0^1 (x^2 - x^3)\ dx = C(1/3 - 1/4) = C(1/12).$$

1b. For $x \in (0,1)$ we have $\int_0^x f_X(t)\ dt = \int_0^x 12t^2(1-t)\ dt = 12(x^3/3 - x^4/4) = 4x^3 - 3x^4$. As such, the cumulative distribution function of X is $F_X(x) := (4x^3 - 3x^4)1_{\{0<x<1\}} + 1_{\{x\geq 1\}}$.

1c. We have $E[X^p] = \int_0^1 x^p\ 12x^2(1-x)\ dx$. If $p \leq -3$, then this integral does not converge and we may say that $E[X^p] = +\infty$. (To see this in detail, note that

$$\int_0^1 x^p\ 12x^2(1-x)\ dx \geq 12\int_0^{1/2} x^{2+p}(1-x)\ dx \geq 6\int_0^{1/2} x^{2+p}\ dx$$

since $x^{2+p}(1-x) \geq 0$ for $x \in (0,1)$ and $(1-x) \geq 1/2$ for $x \in (0,1/2)$. If $p = -3$, then

$$6\int_0^{1/2} x^{2+p}\ dx = 6\log[1/2] - \lim_{x\searrow 0} 6\log[x] = +\infty.$$

If $p < -3$, then

$$6\int_0^{1/2} x^{2+p}\ dx = (1/2)^{3+p}/(3+p) - \lim_{x\searrow 0} x^{3+p}/(3+p) = +\infty.$$

Either way, $E[X^p]$ is bounded below by $+\infty$.) If $p > -3$, then we have

$$\int_0^1 x^p\ 12x^2(1-x)\ dx = 12\int_0^1 x^{2+p}\ dx - 12\int_0^1 x^{3+p}\ dx = 12/(3+p) - 12/(4+p).$$

1d. Put $Y := -\log X$. Since $\mathcal{X} = (0,1)$, we have $\mathcal{Y} = (0,+\infty)$. Then, for $y \in (0,+\infty)$, we have

$$P(Y \leq y) = P(-\log X \leq y) = P(X \geq \exp[-y]) = 1 - P(X \leq \exp[-y]) = 1 - 4\exp[-3y] + 3\exp[-4y].$$

Thus, the cumulative distribution function of Y is $F_Y(y) := (1 - 4\exp[-3y] + 3\exp[-4y])1_{\{y>0\}}$.

1e. For $y \in (0,+\infty)$, we have $\frac{d}{dy}F_Y(y) = 12\exp[-3y](1-\exp[-y])$. For $y \in (-\infty,0)$, we have $\frac{d}{dy}F_Y(y) = 0$. Thus, the probability density function of Y is $f_Y(y) := 12\exp[-3y](1-\exp[-y])1_{\{y>0\}}$.

1f. For $y \in (0,+\infty)$, we have

$$S_Y(y) = 1 - F_Y(y) = (4\exp[-3y] - 3\exp[-4y]).$$

1g. For $y \in (0,+\infty)$, we have

$$h_Y(y) = -\frac{d}{dy}\log[S_Y(y)] = -\frac{d}{dy}\log[4\exp[-3y] - 3\exp[-4y]] = \frac{12\exp[-3y](1-\exp[-y])}{4\exp[-3y] - 3\exp[-4y]} = \frac{12(1-\exp[-y])}{4 - 3\exp[-y]}.$$

1h. Since

$$\lim_{M\to\infty} MS_Y(M) = \lim_{M\to\infty} (4M\exp[-3M] - 3M\exp[-4M]) = 0$$

(L'Hopital's rule shows that, in fact, $\lim_{M\to\infty} M^a \exp[-bM] = 0$ for any positive real numbers a and b), we may calculate $E[Y]$ by integrating $S_Y(y)$ in dy. We have

$$E[Y] = \int_0^\infty S_Y(y)\ dy = \int_0^\infty (4\exp[-3y] - 3\exp[-4y])\ dy = 4/3 - 3/4 = 7/12.$$

2a. We have

$$\exp[tz]\exp[-z^2/2] = \exp[tz - z^2/2] = \exp[-z^2/2 + tz - t^2/2 + t^2/2] = \exp[-(z-t)^2/2 + t^2/2] = \exp[-(z-t)^2/2]\exp[t^2/2].$$

As such,

$$M_Z(t) = E[\exp[tZ]] = \int_{\mathbb{R}} \exp[tz](2\pi)^{-1/2}\exp[-z^2/2]\ dz = \exp[t^2/2]\int_{\mathbb{R}} (2\pi)^{-1/2}\exp[-(z-t)^2/2]\ dz = \exp[t^2/2].$$

(The last integral is 1 because the integrand is the probability density function of a normal random variable with mean t and variance 1.)

2b. We have

$$M_X(t) = E[\exp[tX]] = E[\exp[t(\mu + \sigma Z)]] = E[\exp[t\mu]\exp[t\sigma Z]] = \exp[t\mu]E[\exp[t\sigma Z]] = \exp[t\mu]M_Z(t\sigma).$$

Therefore, we also have

$$M_X(t) = \exp[t\mu]\exp[(t\sigma)^2/2] = \exp[t\mu + t^2\sigma^2/2].$$