BST 675 — Fall 2010 — Dr. Charnigo

Written Assignment 4

Written Assignment 4 is due on Wednesday 03 November at the end of class. You are encouraged to work in groups of two or three, but you may work individually if you prefer.

[30] 1. Consider the mixture model (5) on page 17 of the Unit III notes, and let the events A_1 through A_k be as defined there.

[10] a. Bayes' Theorem would seem to suggest that

$$P(A_1 \mid X = x) = P(X = x \mid A_1)P(A_1) / \sum_{j=1}^{k} P(X = x \mid A_j)P(A_j).$$

Explain, however, why the above formula is problematic.

[20] b. Consider instead

$$P(A_1 \mid x \leq X \leq x + \delta) = P(x \leq X \leq x + \delta \mid A_1)P(A_1) / \sum_{j=1}^{k} P(x \leq X \leq x + \delta \mid A_j)P(A_j),$$

where δ is a positive real number. Explain why, for any $j \in \{1, \ldots, k\}$, we have $\lim_{\delta \searrow 0} P(x \leq X \leq x + \delta \mid A_j)/\delta = (2\pi)^{-1/2}\sigma_j^{-1} \exp[-(x - \mu_j)^2/(2\sigma_j^2)]$. Conclude that

$$\lim_{\delta \searrow 0} P(A_1 \mid x \leq X \leq x + \delta) = \frac{p_1 \sigma_1^{-1} \exp[-(x - \mu_1)^2/(2\sigma_1^2)]}{\sum_{j=1}^{k} p_j \sigma_j^{-1} \exp[-(x - \mu_j)^2/(2\sigma_j^2)]}.$$

Hence, a reasonable definition of $P(A_1 \mid X = x)$ is

$$\frac{p_1 \sigma_1^{-1} \exp[-(x - \mu_1)^2/(2\sigma_1^2)]}{\sum_{j=1}^{k} p_j \sigma_j^{-1} \exp[-(x - \mu_j)^2/(2\sigma_j^2)]}.$$

[20] 2. Let $F(x, y) := y \exp[-(x + y)^2]$.

[10] a. Find $\frac{\partial}{\partial x} F(x, y)$ and $\frac{\partial}{\partial y} F(x, y)$.

[10] b. Find $\frac{\partial^2}{\partial x^2} F(x, y)$, $\frac{\partial^2}{\partial x \partial y} F(x, y)$, and $\frac{\partial^2}{\partial y^2} F(x, y)$.

[20] 3. Let $f(x, y) := \cos(y^2)$ and $R := \{(x, y) : 0 \leq x \leq y \leq 1\}$. Find $\int\int_R f(x, y)\ dx\ dy$.

[30] 4. Let $\mathbf{b} := (-1, 1)^T$, $\mathbf{x} := (x_1, x_2)^T$, and

$$\mathbf{A} := \begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix},$$

where x_1 and x_2 are real numbers.

[10] a. Evaluate $\mathbf{Ab}$.

[10] b. Show that $2|x_1 x_2| \leq x_1^2 + x_2^2$. *Hint*: The result is obviously true if $x_1 x_2 = 0$. If $x_1 x_2 > 0$, then the result claims $x_1^2 + x_2^2 - 2x_1 x_2 \geq 0$. If $x_1 x_2 < 0$, then the result claims $x_1^2 + x_2^2 + 2x_1 x_2 \geq 0$.

[10] c. Evaluate $\mathbf{x}^T \mathbf{A} \mathbf{x}$. Then use part b to show that $\mathbf{A}$ is positive definite.