BST 675 — Fall 2010 — Dr. Charnigo

Solutions to Written Assignment 4

1a. Given that A_j occurs ($j \in \{1, 2, \ldots, k\}$), X is normally distributed with mean μ_j and standard deviation σ_j. As such, for any x we have $P(X = x|A_j) = 0$. Therefore, the proposed formula reduces to 0/0, which we do not know how to interpret.

1b. For any positive real number δ, we have $P(x \le X \le x + \delta|A_j) = F_j(x + \delta) - F_j(x)$, where F_j is defined to be the cumulative distribution function for a normal random variable with mean μ_j and standard deviation σ_j. Since $\lim_{\delta \searrow 0}[F_j(x+\delta) - F_j(x)]/\delta = f_j(x)$, where f_j is defined to be the probability density function for a normal random variable with mean μ_j and standard deviation σ_j, we conclude that $\lim_{\delta \searrow 0} P(x \le X \le x + \delta|A_j)/\delta = f_j(x)$. As such,

$$
\begin{aligned}
\lim_{\delta \searrow 0} P(A_1|x \le X \le x + \delta) &= \lim_{\delta \searrow 0} \frac{P(x \le X \le x + \delta|A_1)P(A_1)}{\sum_{j=1}^{k} P(x \le X \le x + \delta|A_j)P(A_j)} \\
&= \lim_{\delta \searrow 0} \frac{P(x \le X \le x + \delta|A_1)P(A_1)/\delta}{\sum_{j=1}^{k} P(x \le X \le x + \delta|A_j)P(A_j)/\delta} \\
&= \frac{\lim_{\delta \searrow 0} P(x \le X \le x + \delta|A_1)P(A_1)/\delta}{\lim_{\delta \searrow 0} \sum_{j=1}^{k} P(x \le X \le x + \delta|A_j)P(A_j)/\delta} \\
&= \frac{P(A_1) \lim_{\delta \searrow 0} P(x \le X \le x + \delta|A_1)/\delta}{\sum_{j=1}^{k} P(A_j) \lim_{\delta \searrow 0} P(x \le X \le x + \delta|A_j)/\delta} \\
&= \frac{P(A_1) f_1(x)}{\sum_{j=1}^{k} P(A_j) f_j(x)} \\
&= \frac{p_1 (2\pi)^{-1/2} \sigma_1^{-1} \exp[-(x - \mu_1)^2/(2\sigma_1^2)]}{\sum_{j=1}^{k} p_j (2\pi)^{-1/2} \sigma_j^{-1} \exp[-(x - \mu_j)^2/(2\sigma_j^2)]} \\
&= \frac{p_1 \sigma_1^{-1} \exp[-(x - \mu_1)^2/(2\sigma_1^2)]}{\sum_{j=1}^{k} p_j \sigma_j^{-1} \exp[-(x - \mu_j)^2/(2\sigma_j^2)]}.
\end{aligned}
$$

2a. With $F(x, y) = y \exp[-(x + y)^2]$ we have

$$\frac{\partial}{\partial x} F(x, y) = -2y(x + y) \exp[-(x + y)^2]$$

and

$$\frac{\partial}{\partial y} F(x, y) = (1 - 2y(x + y)) \exp[-(x + y)^2].$$

2b. We have

$$\frac{\partial^2}{\partial x^2} F(x, y) = 2(-y + 2y(x + y)^2) \exp[-(x + y)^2],$$

$$\frac{\partial^2}{\partial x \partial y} F(x, y) = 2(-2y - x + 2y(x + y)^2) \exp[-(x + y)^2],$$

and

$$\frac{\partial^2}{\partial y^2} F(x, y) = 2(-3y - 2x + 2y(x + y)^2) \exp[-(x + y)^2].$$

3. We will get stuck if we try to integrate $\cos(y^2)$ in dy, so let us set up the double integral for integration in dx first followed by integration in dy. We have

$$\begin{aligned}
\int\int_R f(x,y)\ dx\ dy &= \int_0^1 \left[\int_0^y \cos(y^2)\ dx\right]\ dy \\
&= \int_0^1 \left[\cos(y^2)\int_0^y 1\ dx\right]\ dy \\
&= \int_0^1 \cos(y^2)\ x|_0^y\ dy \\
&= \int_0^1 \cos(y^2)\ y\ dy \\
&= (1/2)\int_0^1 \cos(y^2)\ 2y\ dy \\
&= (1/2)\sin(y^2)|_0^1 \\
&= (1/2)\sin(1).
\end{aligned}$$

4a. We have

$$\begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} (1\times -1)+(-0.5\times 1) \\ (-0.5\times -1)+(1\times 1) \end{bmatrix} = \begin{bmatrix} -1.5 \\ 1.5 \end{bmatrix}.$$

4b. If $x_1x_2 = 0$, then

$$2|x_1x_2| = 0 \le x_1^2 + x_2^2.$$

If $x_1x_2 > 0$, then

$$2|x_1x_2| = 2x_1x_2 \le (x_1-x_2)^2 + 2x_1x_2 = x_1^2 - 2x_1x_2 + x_2^2 + 2x_1x_2 = x_1^2 + x_2^2.$$

If $x_1x_2 < 0$, then

$$2|x_1x_2| = -2x_1x_2 \le (x_1+x_2)^2 - 2x_1x_2 = x_1^2 + 2x_1x_2 + x_2^2 - 2x_1x_2 = x_1^2 + x_2^2.$$

4c. We have

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} x_1 - 0.5x_2 \\ -0.5x_1 + x_2 \end{bmatrix} = x_1^2 - x_1x_2 + x_2^2.$$

If $x_1x_2 = 0$ (but $\mathbf{x} \neq \mathbf{0}$), then

$$x_1^2 - x_1x_2 + x_2^2 = x_1^2 + x_2^2 > 0.$$

If $x_1x_2 \neq 0$, then

$$x_1x_2 \le |x_1x_2| < 2|x_1x_2| \le x_1^2 + x_2^2, \quad \text{so that} \quad 0 < x_1^2 + x_2^2 - x_1x_2.$$

In either case, we have $x_1^2 - x_1x_2 + x_2^2 > 0$, whence $\mathbf{A}$ is positive definite.