BST 675 — Fall 2010 — Dr. Charnigo

Written Assignment 5 Solutions

1a. We have

$$M_X(t) = \left(\frac{p}{1-(1-p)e^t} \right)^{r_1}$$

and

$$M_Y(t) = \left(\frac{p}{1-(1-p)e^t} \right)^{r_2}$$

for $t < -\log(1-p)$. As such,

$$M_{X+Y}(t) = M_X(t)M_Y(t) = \left(\frac{p}{1-(1-p)e^t} \right)^{r_1+r_2}$$

for $t < -\log(1-p)$. Since $X+Y$ has the same moment generating function in a neighborhood of 0 as a negative binomial random variable with parameters r_1+r_2 and p, the distribution of $X+Y$ is in fact negative binomial with parameters r_1+r_2 and p.

1b. We have

$$M_X(t) = \exp[\lambda_1(e^t-1)]$$

and

$$M_Y(t) = \exp[\lambda_2(e^t-1)].$$

As such,

$$M_{X+Y}(t) = M_X(t)M_Y(t) = \exp[(\lambda_1+\lambda_2)(e^t-1)].$$

Since $X+Y$ has the same moment generating function in a neighborhood of 0 as a Poisson random variable with parameter $\lambda_1+\lambda_2$, the distribution of $X+Y$ is in fact Poisson with parameter $\lambda_1+\lambda_2$.

1c. We have

$$M_X(t) = \exp[\mu_1 t + \sigma_1^2 t^2/2]$$

and

$$M_Y(t) = \exp[\mu_2 t + \sigma_2^2 t^2/2].$$

As such,

$$M_{X+Y}(t) = M_X(t)M_Y(t) = \exp[(\mu_1+\mu_2)t + (\sigma_1^2+\sigma_2^2)t^2/2].$$

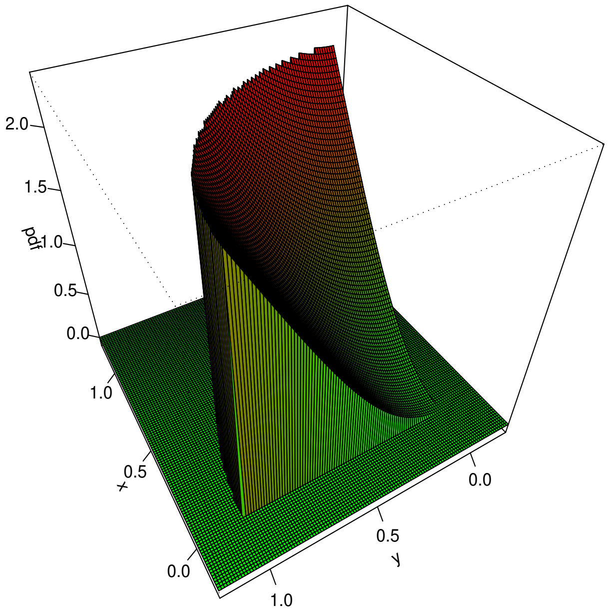
Since $X+Y$ has the same moment generating function in a neighborhood of 0 as a normal random variable with parameters $\mu_1+\mu_2$ and $\sqrt{\sigma_1^2+\sigma_2^2}$, the distribution of $X+Y$ is in fact normal with parameters $\mu_1+\mu_2$ and $\sqrt{\sigma_1^2+\sigma_2^2}$.

2a. The following R code...

```
postscript("C:\\Documents and Settings\\Rich\\Desktop\\Rich\\STA623F10\\WA5Ex2agraph.ps")
C<- 4/(pi*(log(4)-1))
par(bg = "white")
x <- seq(-0.21,1.21, length = 120)
y <- seq(-0.21,1.21, length = 121)
```

```
z <- outer(x, y, function(a,b)
C*log(1+a^2+b^2)*(a>0)*(b>0)*(a^2+b^2<1)
)
nrz <- nrow(z)
ncz <- ncol(z)
jet.colors <- colorRampPalette( c("green", "red") )
nbcol <- 100
color <- jet.colors(nbcol)
zfacet <- z[-1, -1] + z[-1, -ncz] + z[-nrz, -1] + z[-nrz, -ncz]
facetcol <- cut(zfacet, nbcol)
persp(x, y, z, col=color[facetcol], phi=40, theta=-120,zlab="pdf",
xlab="x",ylab="y",axes=T,ticktype="detailed",zlim=c(-0.1,C*log(2)+0.1))
dev.off()
```

...produced the following graphic.



2b. For convenience, put

$$C := \left(\frac{4}{\pi[\log 4 - 1]}\right).$$

Noting that $y > 0$, $x > 0$, and $x^2 + y^2 < 1$ together are equivalent to $0 < x < 1$ and $0 < y < \sqrt{1 - x^2}$, we obtain, for $x \in (0,1)$,

$$
\begin{aligned}
f_X(x) &= C \int_0^{\sqrt{1-x^2}} \log(1+x^2+y^2)\, dy \\
&= C \left\{ y \log(1+x^2+y^2)|_0^{\sqrt{1-x^2}} - 2 \int_0^{\sqrt{1-x^2}} \frac{y^2}{1+x^2+y^2}\, dy \right\} \\
&= C \left\{ \sqrt{1-x^2} \log 2 - 2 \int_0^{\sqrt{1-x^2}} \left(1 - \frac{1+x^2}{1+x^2+y^2}\right) dy \right\} \\
&= C \left\{ \sqrt{1-x^2} \log 2 - 2\sqrt{1-x^2} + 2 \int_0^{\sqrt{1-x^2}} \frac{1+x^2}{1+x^2+y^2}\, dy \right\} \\
&= C \left\{ \sqrt{1-x^2} \log 2 - 2\sqrt{1-x^2} + 2 \int_0^{\sqrt{1-x^2}} \frac{1}{1+(y/\sqrt{1+x^2})^2}\, dy \right\} \\
&= C \left\{ \sqrt{1-x^2}(\log 2 - 2) + 2\sqrt{1+x^2} \arctan\left(y/\sqrt{1+x^2}\right) |_0^{\sqrt{1-x^2}} \right\} \\
&= C \left\{ \sqrt{1-x^2}(\log 2 - 2) + 2\sqrt{1+x^2} \arctan\left(\sqrt{(1-x^2)/(1+x^2)}\right) \right\}.
\end{aligned}
$$

2c. We have, for $x \in (0,1)$,

$$
f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{\log(1+x^2+y^2) 1_{\{0<y<\sqrt{1-x^2}\}}}{\sqrt{1-x^2}(\log 2 - 2) + 2\sqrt{1+x^2} \arctan\left(\sqrt{(1-x^2)/(1+x^2)}\right)}.
$$

2d. Since $f_{X,Y}(x,y) > 0$ for $x > 0$, $y > 0$, and $x^2 + y^2 < 1$, we have

$$
\begin{aligned}
S_{U,V} &= \{(u,v)^T \in \mathbb{R}^2 : \exists (x,y)^T \in \mathbb{R}^2 \text{ such that } u = x^2, v = y^2, f_{X,Y}(x,y) > 0\} \\
&= \{(u,v)^T \in \mathbb{R}^2 : u > 0, v > 0, u + v < 1\}.
\end{aligned}
$$

Put $h_1(u,v) := \sqrt{u}$ and $h_2(u,v) := \sqrt{v}$ for $(u,v)^T \in S_{U,V}$. We have

$$
Det \begin{bmatrix} \frac{\partial h_1(u,v)}{\partial u} & \frac{\partial h_1(u,v)}{\partial v} \\ \frac{\partial h_2(u,v)}{\partial u} & \frac{\partial h_2(u,v)}{\partial v} \end{bmatrix} = Det \begin{bmatrix} (1/2)u^{-1/2} & 0 \\ 0 & (1/2)v^{-1/2} \end{bmatrix} = (1/4)u^{-1/2}v^{-1/2}.
$$

Thus, for $(u,v)^T \in S_{U,V}$,

$$
f_{X,Y}(h_1(u,v), h_2(u,v)) = C \log(1+u+v)
$$

and

$$
f_{U,V}(u,v) = C \log(1+u+v)(1/4)u^{-1/2}v^{-1/2} = \frac{\log(1+u+v)}{\sqrt{uv}\pi(\log 4 - 1)}.
$$

2e. Put $g(x) := x^2$ for $0 < x < 1$. Then, for $0 < u < 1$, $g^{-1}(u) = \sqrt{u}$ and

$$
\begin{aligned}
f_U(u) &= f_X(g^{-1}(u)) \frac{d}{du} g^{-1}(u) \\
&= C \left\{ \sqrt{1-u}(\log 2 - 2) + 2\sqrt{1+u} \arctan\left(\sqrt{(1-u)/(1+u)}\right) \right\} (1/2)\sqrt{u^{-1}} \\
&= (C/2) \left\{ \sqrt{u^{-1}-1}(\log 2 - 2) + 2\sqrt{u^{-1}+1} \arctan\left(\sqrt{(1-u)/(1+u)}\right) \right\} \\
&= \frac{\sqrt{u^{-1}-1}(\log 4 - 4) + 4\sqrt{u^{-1}+1} \arctan\left(\sqrt{(1-u)/(1+u)}\right)}{\pi(\log 4 - 1)}.
\end{aligned}
$$

2f. Since $f_{X,Y}(x,y) > 0$ for $x > 0$, $y > 0$, and $x^2 + y^2 < 1$, we have

$$\begin{aligned} S_{R,\Theta} &= \{(r,\theta)^T \in \mathbb{R}^2 : \exists (x,y)^T \in \mathbb{R}^2 \text{ such that } r = \sqrt{x^2+y^2}, \theta = \arctan(y/x), f_{X,Y}(x,y) > 0\} \\ &= \{(r,\theta)^T \in \mathbb{R}^2 : 0 < r < 1, 0 < \theta < \pi/2\}. \end{aligned}$$

Put $h_1(r,\theta) := r\cos\theta$ and $h_2(r,\theta) := r\sin\theta$ for $(r,\theta)^T \in S_{R,\Theta}$. We have

$$Det \begin{bmatrix} \frac{\partial h_1(r,\theta)}{\partial r} & \frac{\partial h_1(r,\theta)}{\partial \theta} \\ \frac{\partial h_2(r,\theta)}{\partial r} & \frac{\partial h_2(r,\theta)}{\partial \theta} \end{bmatrix} = Det \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = r.$$

Thus, for $(r,\theta)^T \in S_{R,\Theta}$,

$$f_{X,Y}(h_1(r,\theta), h_2(r,\theta)) = C\log(1+r^2)$$

and

$$f_{R,\Theta}(r,\theta) = C\log(1+r^2)r.$$

Note also that

$$f_{R,\Theta}(r,\theta) = \frac{2}{\pi} 1_{\{0<\theta<\pi/2\}} \times \frac{2r}{\log 4 - 1} \log(1+r^2) 1_{\{0<r<1\}} = f_\Theta(\theta) \times f_R(r),$$

so that R and Θ are independent.

2g. (i) For which y the surface $z = f_{X,Y}(x,y)$ lies above the xy-plane depends on x. So the support of Y depends on the value realized by X, implying that X and Y are not independent. (ii) At any fixed $r(= \sqrt{x^2+y^2}) \in (0,1)$, the height of the surface $z = f_{X,Y}(x,y)$ is constant for $\theta(= \arctan(y/x)) \in (0,\pi/2)$. So the distribution of Θ does not depend on the value realized by R (indeed, the distribution is uniform no matter the value realized by R), implying that R and Θ are independent.