BST 675 — Fall 2011 — Dr. Charnigo

Midterm Examination

This non-collaborative take-home midterm examination is due at the end of class on Tuesday 18 October. By non-collaborative I mean that you are not permitted to discuss the examination with anyone other than me, until after the deadline for submission. The examination is to be submitted in hard copy, to me in person or under my office door.

[50] 1. Let X_1 and X_2 have (the same) probability mass function $f(x) := \exp[-\lambda]\lambda^x/x!$ for $x \in \{0, 1, 2, \ldots\}$, where λ is a positive constant. Assume that $\{X_1 = a\}$ and $\{X_2 = b\}$ are independent events for any nonnegative integers a and b.

[20] a. For any nonnegative integer b, show that

$$P(X_1 + X_2 = b) = \sum_{x=0}^{b} P(X_1 + X_2 = b \cap X_2 = x) = \sum_{x=0}^{b} P(X_1 = b - x)P(X_2 = x).$$

[20] b. For any nonnegative integer b, show that

$$\sum_{x=0}^{b} P(X_1 = b - x)P(X_2 = x) = \exp[-2\lambda]\lambda^b \sum_{x=0}^{b} \frac{1}{(b-x)!x!} = \exp[-2\lambda](2\lambda)^b/b!$$

Hint. To establish the second equality, note that $\sum_{x=0}^{b}(1/2)^x(1/2)^{b-x}\binom{b}{x} = 1$ (explain why) and then evaluate $\sum_{x=0}^{b}\binom{b}{x}$.

[10] c. What familiar distribution does $X_1 + X_2$ have ?

[50] 2. Let X have probability density function $f(x) := (1/2)\exp[-|x - \mu|]$, where μ is a real constant.

[20] a. Find $M_X(t)$ for $t \in (-1, 1)$.

Hint. We have

$$\int_{\mathbb{R}} \exp[tx]\exp[-|x-\mu|]\;dx = \int_{-\infty}^{\mu} \exp[tx]\exp[x-\mu]\;dx + \int_{\mu}^{\infty} \exp[tx]\exp[\mu - x]\;dx.$$

[10] b. What is the expected value of X ?

[20] c. Find the probability density function of $Y := |X - \mu|$.

Hint. First find the cumulative distribution function of Y by noting that, for any nonnegative real y, the event $\{Y \le y\}$ can be expressed in the form $\{a(y) \le X \le b(y)\}$. Then note that $\frac{d}{dy}\int_{a(y)}^{b(y)} f(x)\;dx = f(b(y))b'(y) - f(a(y))a'(y)$, so that solving this problem does not actually require you to find an explicit formula for the cumulative distribution function of X.