BST 675 — Fall 2011 — Dr. Charnigo

Written Assignment 2

Written Assignment 2 is due on Tuesday 04 October at the end of class. You are encouraged to work in groups of two or three, but you may work individually if you prefer.

Let X be an offset geometric random variable with probability mass function $f(x) := p(1-p)^{x-1}1_{\{x \in \{1,2,\ldots\}\}}$, where $p \in (0,1)$.

[10] 1. Show that

$$\sum_{x=1}^{\infty} x^2 f(x) = \sum_{x=1}^{\infty} p(1-p)^{x-1} + (2^2-1^2)\sum_{x=2}^{\infty} p(1-p)^{x-1} + (3^2-2^2)\sum_{x=3}^{\infty} p(1-p)^{x-1} + \cdots.$$

[10] 2. Show that

$$\sum_{x=1}^{\infty} p(1-p)^{x-1} + (2^2-1^2)\sum_{x=2}^{\infty} p(1-p)^{x-1} + (3^2-2^2)\sum_{x=3}^{\infty} p(1-p)^{x-1} + \cdots = 1 + 3(1-p) + 5(1-p)^2 + \cdots.$$

[10] 3. Show that

$$1 + 3(1-p) + 5(1-p)^2 + \cdots = \sum_{x=0}^{\infty}(1-p)^x + 2\sum_{x=1}^{\infty}(1-p)^x + 2\sum_{x=2}^{\infty}(1-p)^x + \cdots.$$

[10] 4. Show that

$$\sum_{x=0}^{\infty}(1-p)^x + 2\sum_{x=1}^{\infty}(1-p)^x + 2\sum_{x=2}^{\infty}(1-p)^x + \cdots = 1/p + 2(1-p)/p + 2(1-p)^2/p + \cdots.$$

[10] 5. Show that

$$1/p + 2(1-p)/p + 2(1-p)^2/p + \cdots = -1/p + \{1 + (1-p) + (1-p)^2 + \cdots\}2/p.$$

[10] 6. Show that

$$-1/p + \{1 + (1-p) + (1-p)^2 + \cdots\}2/p = -1/p + 2/p^2.$$

Remark: Items 1 through 6 show that $E[X^2] = 2/p^2 - 1/p$.

[10] 7. We have $1/p = \sum_{x=1}^{\infty}(1-p)^{x-1}$. Assuming that term-by-term differentiation is legal, show that

$$1/p^2 = \sum_{x=2}^{\infty}(x-1)(1-p)^{x-2} = \sum_{y=1}^{\infty} y(1-p)^{y-1}.$$

Conclude that $E[X] = 1/p$.

[10] 8. Now show that

$$2/p^3 = \sum_{x=3}^{\infty}(x-1)(x-2)(1-p)^{x-3} = \sum_{y=1}^{\infty}(y+1)y(1-p)^{y-1}.$$

Conclude that $2/p^2 = E[X^2 + X]$ and, hence, $E[X^2] = 2/p^2 - 1/p$. *Remark:* Items 7 and 8 show, more easily than items 1 through 6, that $E[X^2] = 2/p^2 - 1/p$.

[10] 9. Obtain the moment generating function $M_X(t)$.

[10] 10. Calculate $M_X''(t)$ and evaluate it at $t = 0$. *Remark:* Items 9 and 10 show, more easily than items 1 through 6, that $E[X^2] = 2/p^2 - 1/p$.