BST 675 — Fall 2011 — Dr. Charnigo

Written Assignment 2 Solutions

1. We have

$$\begin{aligned}
\sum_{x=1}^{\infty} x^2 f(x) &= 1^2 p(1-p)^0 + 2^2 p(1-p)^1 + 3^2 p(1-p)^2 + \cdots \\
&= 1^2 p\{(1-p)^0 + (1-p)^1 + (1-p)^2 + \cdots\} \\
&+ (2^2 - 1^2) p\{(1-p)^1 + (1-p)^2 + (1-p)^3 + \cdots\} \\
&+ (3^2 - 2^2) p\{(1-p)^2 + (1-p)^3 + (1-p)^4 + \cdots\} + \cdots \\
&= \sum_{x=1}^{\infty} p(1-p)^{x-1} + (2^2 - 1^2) \sum_{x=2}^{\infty} p(1-p)^{x-1} + (3^2 - 2^2) \sum_{x=3}^{\infty} p(1-p)^{x-1} + \cdots .
\end{aligned}$$

2. We have

$$\begin{aligned}
& \sum_{x=1}^{\infty} p(1-p)^{x-1} + (2^2 - 1^2) \sum_{x=2}^{\infty} p(1-p)^{x-1} + (3^2 - 2^2) \sum_{x=3}^{\infty} p(1-p)^{x-1} + \cdots \\
= & \frac{p}{1-(1-p)} + (2^2 - 1^2) \frac{p(1-p)}{1-(1-p)} + (3^2 - 2^2) \frac{p(1-p)^2}{1-(1-p)} + \cdots \\
= & 1 + 3(1-p) + 5(1-p)^2 + \cdots .
\end{aligned}$$

3. We have

$$\begin{aligned}
& 1 + 3(1-p) + 5(1-p)^2 + \cdots \\
= & 1 + (1-p) + (1-p)^2 + \cdots \\
+ & 2(1-p) + 2(1-p)^2 + 2(1-p)^3 + \cdots \\
+ & 2(1-p)^2 + 2(1-p)^3 + 2(1-p)^4 + \cdots \\
= & \sum_{x=0}^{\infty} (1-p)^x + 2 \sum_{x=1}^{\infty} (1-p)^x + 2 \sum_{x=2}^{\infty} (1-p)^x + \cdots .
\end{aligned}$$

4. We have

$$\begin{aligned}
& \sum_{x=0}^{\infty} (1-p)^x + 2 \sum_{x=1}^{\infty} (1-p)^x + 2 \sum_{x=2}^{\infty} (1-p)^x + \cdots \\
= & \frac{1}{1-(1-p)} + \frac{2(1-p)}{1-(1-p)} + \frac{2(1-p)^2}{1-(1-p)} + \cdots \\
= & 1/p + 2(1-p)/p + 2(1-p)^2/p + \cdots .
\end{aligned}$$

5. We have

$$\begin{aligned}
& 1/p + 2(1-p)/p + 2(1-p)^2/p + \cdots \\
= & -1/p + 2/p + 2(1-p)/p + 2(1-p)^2/p + \cdots \\
= & -1/p + \{1 + (1-p) + (1-p)^2 + \cdots\} 2/p.
\end{aligned}$$

6. We have

$$\begin{aligned} & -1/p + \{1 + (1-p) + (1-p)^2 + \cdots\}2/p \\ = & -1/p + \frac{1}{1-(1-p)}2/p \\ = & -1/p + (1/p)(2/p) \\ = & -1/p + 2/p^2. \end{aligned}$$

7. We have

$$1/p = \sum_{x=1}^{\infty} (1-p)^{x-1}.$$

Assuming that term-by-term differentiation is legal, we have

$$\frac{d}{dp}(1/p) = \sum_{x=1}^{\infty} \frac{\partial}{\partial p}(1-p)^{x-1}$$

or

$$-1/p^2 = \sum_{x=1}^{\infty} -(x-1)(1-p)^{x-2} = \sum_{x=2}^{\infty} -(x-1)(1-p)^{x-2}.$$

Multiplying through by -1 and putting $y := x - 1$, we obtain

$$1/p^2 = \sum_{y=1}^{\infty} y(1-p)^{y-1}.$$

8. We have

$$1/p^2 = \sum_{x=2}^{\infty} (x-1)(1-p)^{x-2}.$$

Differentiating term-by-term, we have

$$\frac{d}{dp}(1/p^2) = \sum_{x=2}^{\infty} \frac{\partial}{\partial p}(x-1)(1-p)^{x-2}$$

or

$$-2/p^3 = \sum_{x=2}^{\infty} -(x-1)(x-2)(1-p)^{x-3} = \sum_{x=3}^{\infty} -(x-1)(x-2)(1-p)^{x-3}.$$

Multiplying through by -1 and putting $y := x - 2$, we obtain

$$2/p^3 = \sum_{y=1}^{\infty} (y+1)y(1-p)^{y-1}.$$

We have

$$E[X^2+X] = \sum_{x=1}^{\infty} (x^2+x)f(x) = \sum_{x=1}^{\infty} (x+1)xp(1-p)^{x-1} = p\sum_{x=1}^{\infty} (x+1)x(1-p)^{x-1} = p(2/p^3) = 2/p^2.$$

As such,

$$E[X^2] = E[X^2 + X] - E[X] = 2/p^2 - 1/p.$$

(To see that $E[X] = 1/p$, multiply by p both sides of the equality obtained in exercise 7.)

9. We have

$$\begin{aligned} M_X(t) &= E[\exp(tX)] \\ &= \sum_{x=1}^{\infty} \exp(tx) f(x) \\ &= \sum_{x=1}^{\infty} \exp(tx) p(1-p)^{x-1} \\ &= p\exp(t) \sum_{x=1}^{\infty} \{(1-p)\exp(t)\}^{x-1} \\ &= \frac{p\exp(t)}{1-(1-p)\exp(t)} = \frac{p}{\exp(-t)-(1-p)} \end{aligned}$$

for $t < -\log(1-p)$.

10. We have

$$M_X'(t) = \frac{p}{\{\exp(-t)-(1-p)\}^2} \exp(-t)$$

and

$$M_X''(t) = \frac{-p}{\{\exp(-t)-(1-p)\}^2} \exp(-t) + \frac{2p}{\{\exp(-t)-(1-p)\}^3} \exp(-2t).$$

Evaluating the latter at 0 yields

$$\frac{-p}{\{1-(1-p)\}^2} + \frac{2p}{\{1-(1-p)\}^3} = \frac{-p}{p^2} + \frac{2p}{p^3} = -1/p + 2/p^2.$$