BST 675 — Fall 2011 — Dr. Charnigo

Written Assignment 3 Solutions

1a. We have

$$\begin{aligned} M_{Z_n}(t) &= E[\exp(tZ_n)] \\ &= E[\exp(t(2n)^{-1/2}X_n)]\exp(-tn(2n)^{-1/2}) \\ &= M_{X_n}(t(2n)^{-1/2})\exp(-t\sqrt{n/2}) \\ &= (1-t\sqrt{2/n})^{-n/2}\exp(-t\sqrt{n/2}) \end{aligned}$$

for $t < \sqrt{n/2}$.

1b. In what follows, suppose that $|t| \in (0, 1/2]$. Note that $\exp(-t\sqrt{n/2}) = \exp(t\sqrt{2/n})^{-n/2}$ and

$$\begin{aligned} (1-t\sqrt{2/n})\exp(t\sqrt{2/n}) &= (1-t\sqrt{2/n})/\exp(-t\sqrt{2/n}) \\ &= 1-(\exp(-t\sqrt{2/n})-1+t\sqrt{2/n})/\exp(-t\sqrt{2/n}) \\ &= 1-(\{1+o(1)\}t^2/n)/\{1+o(1)\} \\ &= 1-(\{1+o(1)\}t^2/n), \end{aligned}$$

where $o(1) \to 0$ as $t\sqrt{2/n} \to 0$. Above, we have used the facts that $\exp(-u) = 1-u+\{1+o(1)\}u^2/2$ and $\exp(-u) = \{1+o(1)\}$ as $u \to 0$. Thus, we have

$$\begin{aligned} M_{Z_n}(t) &= [(1-t\sqrt{2/n})\exp(t\sqrt{2/n})]^{-n/2} \\ &= [1-(\{1+o(1)\}t^2/n)]^{-n/2} \\ &= ([1-(\{1+o(1)\}t^2/n)]^n)^{-1/2} \\ &\to \exp[-t^2]^{-1/2} \\ &= \exp[t^2/2]. \end{aligned}$$

Above, we have used the fact that $(1+a\{1+o(1)\}/n)^n \to \exp[a]$ as $n \to \infty$.

1c. Since $M_{Z_n}(t) \to M_Z(t)$ for all $t \in [-1/2, 1/2]$, where Z has the standard normal distribution, we have

$$P(X_n \le z(2n)^{1/2}+n) = P(Z_n \le z) \to P(Z \le z) = \int_{-\infty}^{z}(2\pi)^{-1/2}\exp[-t^2/2]\,dt.$$

1d. The chi-square distribution on n degrees of freedom is approximately bell-shaped when n is large.

1e. The approximate probabilities are

$$\begin{aligned} P(180 \le X_{200} \le 220) &= P((180-200)/20 \le (X_{200}-200)/20 \le (220-200)/20) \\ &= P(-1 \le (X_{200}-200)/20 \le 1) \\ &\approx P(-1 \le Z \le 1) = 0.6827. \end{aligned}$$

Similarly, $P(0 \le X_2 \le 4) \approx 0.6827$. On the other hand, the actual probabilities are (as determined by software) $P(180 \le X_{200} \le 220) = 0.6835$ and $P(0 \le X_2 \le 4) = 0.8647$.

2a. Let $F(x, y) := (xy)^2 \sin[(xy)^2]$. We have

$$\frac{\partial}{\partial x} F(x, y) = 2(xy)y \sin[(xy)^2] + (xy)^2 \cos[(xy)^2]2(xy)y = 2xy^2 \sin[(xy)^2] + 2x^3y^4 \cos[(xy)^2]$$

and, similarly,

$$\frac{\partial}{\partial y} F(x, y) = 2(xy)x \sin[(xy)^2] + (xy)^2 \cos[(xy)^2]2(xy)x = 2x^2y \sin[(xy)^2] + 2x^4y^3 \cos[(xy)^2].$$

2b. We have

$$\begin{aligned}
\frac{\partial^2}{\partial x^2} F(x, y) &= 2y^2 \sin[(xy)^2] + 2xy^2 \cos[(xy)^2]2(xy)y + 6x^2y^4 \cos[(xy)^2] - 2x^3y^4 \sin[(xy)^2]2(xy)y \\
&= (2y^2 - 4x^4y^6) \sin[(xy)^2] + 10x^2y^4 \cos[(xy)^2],
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2}{\partial x \partial y} F(x, y) &= 4xy \sin[(xy)^2] + 2(xy)y \cos[(xy)^2]2(xy)x + 8x^3y^3 \cos[(xy)^2] - 2x^3y^4 \sin[(xy)^2]2(xy)x \\
&= (4xy - 4x^5y^5) \sin[(xy)^2] + 12x^3y^3 \cos[(xy)^2],
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2}{\partial y^2} F(x, y) &= 2x^2 \sin[(xy)^2] + 2x^2y \cos[(xy)^2]2(xy)x + 6x^4y^2 \cos[(xy)^2] - 2x^4y^3 \sin[(xy)^2]2(xy)x \\
&= (2x^2 - 4x^6y^4) \sin[(xy)^2] + 10x^4y^2 \cos[(xy)^2].
\end{aligned}$$

3. Let $f(x, y) := \exp(-x^2)$ and $R := \{(x, y)^T : 0 \le x \le y \le 1\}$. Putting $\sigma^2 := 1/2$ and letting Y denote a normal random variable with mean 0 and variance σ^2, we have

$$\begin{aligned}
\int \int_R f(x, y)\ dx\ dy &= \int_0^1 \left\{ \int_x^1 \exp(-x^2)\ dy \right\}\ dx \\
&= \int_0^1 \exp(-x^2)(1 - x)\ dx \\
&= \int_0^1 \exp(-x^2)\ dx + (1/2) \int_0^1 \exp(-x^2)(-2x)\ dx \\
&= (2\pi\sigma^2)^{1/2} \int_0^1 (2\pi\sigma^2)^{-1/2} \exp(-x^2/(2\sigma^2))\ dx + (1/2) \int_0^{-1} \exp(u)\ du \\
&= (2\pi\sigma^2)^{1/2} P(0 \le Y \le 1) + (1/2)[\exp(-1) - 1] \\
&= \pi^{1/2} P(0 \le Y/\sigma \le 1/\sigma) + (1/2)[\exp(-1) - 1] \\
&= \pi^{1/2} \{\Phi[1/\sigma] - \Phi[0]\} + (1/2)[\exp(-1) - 1] \\
&= \pi^{1/2} \{\Phi[2^{1/2}] - 1/2\} + (1/2)[\exp(-1) - 1] \\
&= \pi^{1/2} \Phi[2^{1/2}] + (1/2)[\exp(-1) - 1 - \pi^{1/2}],
\end{aligned}$$

where $\Phi[\cdot]$ is the standard normal cumulative distribution function.

4a. Let $\mathbf{b} := (-1, 4)^T$, $\mathbf{x} := (x_1, x_2)^T$, and

$$\mathbf{A} := \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix},$$

where x_1 and x_2 are real numbers. We have

$$\begin{aligned} \mathbf{Ab} &= \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} (3 \times -1) + (1 \times 4) \\ (1 \times -1) + (2 \times 4) \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 7 \end{bmatrix}. \end{aligned}$$

4b. We have

$$\begin{aligned} \mathbf{x}^T \mathbf{A} \mathbf{x} &= (x_1, x_2)^T \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= (x_1, x_2)^T \begin{bmatrix} (3 \times x_1) + (1 \times x_2) \\ (1 \times x_1) + (2 \times x_2) \end{bmatrix} \\ &= 3x_1^2 + 2x_1x_2 + 2x_2^2 \\ &= 2x_1^2 + x_2^2 + (x_1^2 + x_2^2 + 2x_1x_2) \\ &= 2x_1^2 + x_2^2 + (x_1 + x_2)^2 \\ &= c_1x_1^2 + c_2x_2^2 + c_3(x_1 + x_2)^2, \end{aligned}$$

where $c_1 := 2$, $c_2 := 1$, and $c_3 := 1$. Since $2x_1^2 + x_2^2 + (x_1 + x_2)^2$ is a sum of nonnegative quantities, $2x_1^2 + x_2^2 + (x_1 + x_2)^2$ is itself nonnegative. Moreover, $2x_1^2 + x_2^2 + (x_1 + x_2)^2$ can only be zero if $x_1 = x_2 = 0$, establishing that $\mathbf{x}^T \mathbf{A} \mathbf{x} > 0$ whenever $\mathbf{x} \neq \mathbf{0}$ (i.e., $\mathbf{A}$ is positive definite).