BST 675 — Fall 2011 — Dr. Charnigo

Written Assignment 3 Solutions

1a. We have

$$\begin{aligned}
M_{Z_n}(t) &= E[\exp(tZ_n)] \\
&= E[\exp(t(2n)^{-1/2}X_n)]\exp(-tn(2n)^{-1/2}) \\
&= M_{X_n}(t(2n)^{-1/2})\exp(-t\sqrt{n/2}) \\
&= (1-t\sqrt{2/n})^{-n/2}\exp(-t\sqrt{n/2})
\end{aligned}$$

for $t < \sqrt{n/2}$.

1b. In what follows, suppose that $|t| \in (0, 1/2]$. Note that $\exp(-t\sqrt{n/2}) = \exp(t\sqrt{2/n})^{-n/2}$ and

$$\begin{aligned}
(1-t\sqrt{2/n})\exp(t\sqrt{2/n}) &= (1-t\sqrt{2/n})/\exp(-t\sqrt{2/n}) \\
&= 1-(\exp(-t\sqrt{2/n})-1+t\sqrt{2/n})/\exp(-t\sqrt{2/n}) \\
&= 1-(\{1+o(1)\}t^2/n)/\{1+o(1)\} \\
&= 1-(\{1+o(1)\}t^2/n),
\end{aligned}$$

where $o(1) \to 0$ as $t\sqrt{2/n} \to 0$. Above, we have used the facts that $\exp(-u) = 1-u+\{1+o(1)\}u^2/2$ and $\exp(-u) = \{1+o(1)\}$ as $u \to 0$. Thus, we have

$$\begin{aligned}
M_{Z_n}(t) &= [(1-t\sqrt{2/n})\exp(t\sqrt{2/n})]^{-n/2} \\
&= [1-(\{1+o(1)\}t^2/n)]^{-n/2} \\
&= ([1-(\{1+o(1)\}t^2/n)]^n)^{-1/2} \\
&\to \exp[-t^2]^{-1/2} \\
&= \exp[t^2/2].
\end{aligned}$$

Above, we have used the fact that $(1+a\{1+o(1)\}/n)^n \to \exp[a]$ as $n \to \infty$.

1c. Since $M_{Z_n}(t) \to M_Z(t)$ for all $t \in [-1/2, 1/2]$, where Z has the standard normal distribution, we have

$$P(X_n \leq z(2n)^{1/2}+n) = P(Z_n \leq z) \to P(Z \leq z) = \int_{-\infty}^{z} (2\pi)^{-1/2}\exp[-t^2/2]\; dt.$$

1d. The chi-square distribution on n degrees of freedom is approximately bell-shaped when n is large.

1e. The approximate probabilities are

$$\begin{aligned}
P(180 \leq X_{200} \leq 220) &= P((180-200)/20 \leq (X_{200}-200)/20 \leq (220-200)/20) \\
&= P(-1 \leq (X_{200}-200)/20 \leq 1) \\
&\approx P(-1 \leq Z \leq 1) = 0.6827.
\end{aligned}$$

Similarly, $P(0 \leq X_2 \leq 4) \approx 0.6827$. On the other hand, the actual probabilities are (as determined by software) $P(180 \leq X_{200} \leq 220) = 0.6835$ and $P(0 \leq X_2 \leq 4) = 0.8647$.

2a. Let $F(x,y) := (xy)^2 \sin[(xy)^2]$. We have

$$\frac{\partial}{\partial x}F(x,y) = 2(xy)y\sin[(xy)^2] + (xy)^2\cos[(xy)^2]2(xy)y = 2xy^2\sin[(xy)^2] + 2x^3y^4\cos[(xy)^2]$$

and, similarly,

$$\frac{\partial}{\partial y}F(x,y) = 2(xy)x\sin[(xy)^2] + (xy)^2\cos[(xy)^2]2(xy)x = 2x^2y\sin[(xy)^2] + 2x^4y^3\cos[(xy)^2].$$

2b. We have

$$\begin{aligned}\frac{\partial^2}{\partial x^2}F(x,y) &= 2y^2\sin[(xy)^2] + 2xy^2\cos[(xy)^2]2(xy)y + 6x^2y^4\cos[(xy)^2] - 2x^3y^4\sin[(xy)^2]2(xy)y \\ &= (2y^2 - 4x^4y^6)\sin[(xy)^2] + 10x^2y^4\cos[(xy)^2],\end{aligned}$$

$$\begin{aligned}\frac{\partial^2}{\partial x\partial y}F(x,y) &= 4xy\sin[(xy)^2] + 2(xy)y\cos[(xy)^2]2(xy)x + 8x^3y^3\cos[(xy)^2] - 2x^3y^4\sin[(xy)^2]2(xy)x \\ &= (4xy - 4x^5y^5)\sin[(xy)^2] + 12x^3y^3\cos[(xy)^2],\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2}{\partial y^2}F(x,y) &= 2x^2\sin[(xy)^2] + 2x^2y\cos[(xy)^2]2(xy)x + 6x^4y^2\cos[(xy)^2] - 2x^4y^3\sin[(xy)^2]2(xy)x \\ &= (2x^2 - 4x^6y^4)\sin[(xy)^2] + 10x^4y^2\cos[(xy)^2].\end{aligned}$$

3. Let $f(x,y) := \exp(-x^2)$ and $R := \{(x,y)^T : 0 \le x \le y \le 1\}$. Putting $\sigma^2 := 1/2$ and letting Y denote a normal random variable with mean 0 and variance σ^2, we have

$$\begin{aligned}\int\int_R f(x,y)\ dx\ dy &= \int_0^1 \left\{\int_x^1 \exp(-x^2)\ dy\right\}\ dx \\ &= \int_0^1 \exp(-x^2)(1-x)\ dx \\ &= \int_0^1 \exp(-x^2)\ dx + (1/2)\int_0^1 \exp(-x^2)(-2x)\ dx \\ &= (2\pi\sigma^2)^{1/2}\int_0^1 (2\pi\sigma^2)^{-1/2}\exp(-x^2/(2\sigma^2))\ dx + (1/2)\int_0^{-1}\exp(u)\ du \\ &= (2\pi\sigma^2)^{1/2}P(0 \le Y \le 1) + (1/2)[\exp(-1) - 1] \\ &= \pi^{1/2}P(0 \le Y/\sigma \le 1/\sigma) + (1/2)[\exp(-1) - 1] \\ &= \pi^{1/2}\{\Phi[1/\sigma] - \Phi[0]\} + (1/2)[\exp(-1) - 1] \\ &= \pi^{1/2}\{\Phi[2^{1/2}] - 1/2\} + (1/2)[\exp(-1) - 1] \\ &= \pi^{1/2}\Phi[2^{1/2}] + (1/2)[\exp(-1) - 1 - \pi^{1/2}],\end{aligned}$$

where $\Phi[\cdot]$ is the standard normal cumulative distribution function.

4a. Let $\mathbf{b} := (-1, 4)^T$, $\mathbf{x} := (x_1, x_2)^T$, and

$$\mathbf{A} := \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix},$$

where x_1 and x_2 are real numbers. We have

$$\begin{aligned} \mathbf{Ab} &= \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} (3 \times -1) + (1 \times 4) \\ (1 \times -1) + (2 \times 4) \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 7 \end{bmatrix}. \end{aligned}$$

4b. We have

$$\begin{aligned} \mathbf{x}^T\mathbf{Ax} &= (x_1, x_2)^T \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= (x_1, x_2)^T \begin{bmatrix} (3 \times x_1) + (1 \times x_2) \\ (1 \times x_1) + (2 \times x_2) \end{bmatrix} \\ &= 3x_1^2 + 2x_1x_2 + 2x_2^2 \\ &= 2x_1^2 + x_2^2 + (x_1^2 + x_2^2 + 2x_1x_2) \\ &= 2x_1^2 + x_2^2 + (x_1 + x_2)^2 \\ &= c_1x_1^2 + c_2x_2^2 + c_3(x_1 + x_2)^2, \end{aligned}$$

where $c_1 := 2$, $c_2 := 1$, and $c_3 := 1$. Since $2x_1^2 + x_2^2 + (x_1 + x_2)^2$ is a sum of nonnegative quantities, $2x_1^2 + x_2^2 + (x_1 + x_2)^2$ is itself nonnegative. Moreover, $2x_1^2 + x_2^2 + (x_1 + x_2)^2$ can only be zero if $x_1 = x_2 = 0$, establishing that $\mathbf{x}^T\mathbf{Ax} > 0$ whenever $\mathbf{x} \neq \mathbf{0}$ (i.e., $\mathbf{A}$ is positive definite).