BST 676 — Spring 2010 — Dr. Charnigo

Final Examination

The Final Examination, a strictly individual activity, may be submitted any time between 12 Noon on Monday 03 May and 5 p.m. on Wednesday 05 May.

[50] 1. Suppose that $X_1, \ldots, X_n \overset{iid}{\sim} f(x;\theta) := \theta x^{\theta-1}$ for $x \in (0,1)$ and $\theta \in \Theta := (0,\infty)$. Let θ_0 be a fixed element of Θ. Written Assignment 5 showed that a uniformly most powerful test of $H_0 : \theta \geq \theta_0$ against $H_1 : \theta < \theta_0$ entails rejecting the null hypothesis if $-\sum_{i=1}^n \log X_i > g_{n,1,1-\alpha}/\theta_0$. Similarly, a uniformly most powerful test of $H_0 : \theta \leq \theta_0$ against $H_1 : \theta > \theta_0$ entails rejecting the null hypothesis if $-\sum_{i=1}^n \log X_i < g_{n,1,\alpha}/\theta_0$.

[10] a. Find the uniformly most accurate interval for θ among all $100(1-\alpha)\%$ intervals of the form $(0, U(\mathbf{X})]$.

[10] b. Find the uniformly most accurate interval for θ among all $100(1-\alpha)\%$ intervals of the form $[L(\mathbf{X}), \infty)$.

[10] c. Suppose that we wish to create a two-sided $100(1-\alpha)\%$ interval for θ, starting from the inequality $a < -\theta \sum_{i=1}^n \log X_i < b$. What condition must be placed on a and b?

[10] d. Suppose, moreover, that we wish to have the shortest possible two-sided $100(1-\alpha)\%$ interval for θ, starting from the inequality $a < -\theta \sum_{i=1}^n \log X_i < b$. What additional condition must be placed on a and b, besides that identified in part c?

[10] e. With the aid of statistical software, find the shortest possible two-sided 95% interval when $n = 8$.

[20] 2. Work out the computational details for the second strategy on page 12 of Unit VI.

[10] a. Find c such that $P((T/4)^4 \exp[-T+4] < c) = 0.05$. Two possible ways to do this are as follows.

(i) Simulation: Generate 10000 realizations of T. Associated with each is a realization of $U := (T/4)^4 \exp[-T+4]$. Using the 10000 realizations of U, estimate the 5^{th} percentile of its distribution.

(ii) Analysis: Make an initial guess for c. By trial and error or by a numerical method for solving equations, find the numbers a and b, with $a < b$, such that $(a/4)^4 \exp[-a+4] = (b/4)^4 \exp[-b+4] = c$. Then $(T/4)^4 \exp[-T+4] < c$ has the same probability as $T \notin [a,b]$, which can be calculated with the aid of statistical software. If the probability exceeds 0.05, your next guess for c should be lower. If the probability is less than 0.05, your next guess for c should be higher.

[10] b. Suppose, for instance, that $\bar{X} = 3.44$. By trial and error or by a numerical method for solving equations, identify the two values of θ_0 for which $(\bar{X}/\theta_0)^4 \exp[-(X_1 + X_2 + X_3 + X_4)/\theta_0 + 4] = c$. These are the endpoints of the 95% confidence interval for θ obtained by inverting the likelihood ratio test statistic, if we observe $\bar{X} = 3.44$.

[30] 3. Suppose that $X_1, \ldots, X_n \overset{iid}{\sim} Pois(\theta)$ for $\theta \in \Theta := (0, \infty)$. The Mock Comprehensive Examination showed that an approximate $100(1-\alpha)\%$ Wald interval for θ is $\bar{X} \pm z_{1-\alpha/2}\sqrt{\bar{X}/n}$.

[10] a. Show that the approximate $100(1-\alpha)\%$ Wald interval fits expression (21) from Unit VII with $T := \sum_{i=1}^{n} X_i$.

[10] b. Find numbers L and U, depending on θ, α, and n but not on the observed data, such that $\bar{X} - z_{1-\alpha/2}\sqrt{\bar{X}/n} \leq \theta \leq \bar{X} + z_{1-\alpha/2}\sqrt{\bar{X}/n}$ is equivalent to $L \leq T \leq U$.

[10] c. Suppose, for instance, that $n = 10$ and $\theta = 2$. What is the actual confidence level of the approximate 95% Wald interval?