# BST 676 — Spring 2010 — Dr. Charnigo

## Written Assignment 3 Solutions

1. We have

$$\int_0^1 (-\log x)\theta x^{\theta-1}\ dx = \int_0^\infty w\theta \exp[-w\theta]\ dw = -w\exp[-w\theta]|_0^\infty + \int_0^\infty \exp[-w\theta]\ dw = 0 + 1/\theta = 1/\theta$$

via the substitution $w := -\log x$, $dw := -dx/x$ and integration by parts with $u := w$, $dv := \theta\exp[-w\theta]\ dw$, $du := dw$, $v := -\exp[-w\theta]$.

Similarly, we have

$$\int_0^1 (-\log x)^2\theta x^{\theta-1}\ dx = \int_0^\infty w^2\theta \exp[-w\theta]\ dw = (2/\theta)\int_0^\infty w\theta\exp[-w\theta]\ dw = 2/\theta^2.$$

2. For $x \in (0,1)$ and $\theta \in (0,\infty)$ we have

$$\log f(x;\theta) = \log\theta + (\theta-1)\log x, \quad \frac{\partial \log f(x;\theta)}{\partial\theta} = 1/\theta + \log x, \quad \text{and} \quad \frac{\partial^2 \log f(x;\theta)}{\partial\theta^2} = -1/\theta^2.$$

As such, $J_n(\theta) = -nE_\theta[-1/\theta^2] = n/\theta^2$, yielding a Cramer-Rao lower bound of

$$\frac{\tau'(\theta)^2}{J_n(\theta)} = \frac{1/\theta^4}{n/\theta^2} = \frac{1}{n\theta^2}$$

for unbiased estimation of $\tau(\theta) := 1/\theta$.

3. We have

$$E_\theta[-\log X_1] = 1/\theta \quad \text{and} \quad Var_\theta[-\log X_1] = E_\theta[(-\log X_1)^2] - (E_\theta[-\log X_1])^2 = 1/\theta^2,$$

so that

$$\hat{\tau} := -n^{-1}\sum_{i=1}^n \log X_i$$

has expected value $1/\theta$ and variance $1/(n\theta^2)$. Since $\hat{\tau}$ is unbiased for $1/\theta$ and its variance attains the Cramer-Rao lower bound, we conclude that $\hat{\tau}$ is the best unbiased estimator of $1/\theta$.

4. We have

$$E_\theta[a\hat{\tau}] = a/\theta = 1/\theta - (1-a)/\theta \quad \text{and} \quad Var_\theta[a\hat{\tau}] = a^2/(n\theta^2).$$

As such,

$$MSE_\theta[a\hat{\tau}] = \frac{(1-a)^2 + a^2/n}{\theta^2}.$$

Differentiating $(1-a)^2 + a^2/n$ with respect to $a$ and setting the result to zero yields $-2(1-a) + 2a/n = 0$ with unique solution $a = n/(n+1)$. Since $-2(1-a)+2a/n < 0$ for $a < n/(n+1)$ and $-2(1-a)+2a/n > 0$ for $a > n/(n+1)$, we conclude that $a = n/(n+1)$ in fact minimizes the mean square error.

5. Choosing $a = 1$ minimizes criterion (13). In fact, I will prove that this is a general result, not just applicable to data governed by a beta distribution. Let $W(\mathbf{X})$ be any unbiased estimator of any

function $\lambda(\theta)$ such that $\log[W(\mathbf{X})/\lambda(\theta)]$ has finite expectation. Consider estimators of the form $aW(\mathbf{X})$ for $a \in (0, \infty)$. We have

$$E_\theta[aW(\mathbf{X})/\lambda(\theta) - 1 - \log[aW(\mathbf{X})/\lambda(\theta)]] = a - 1 - \log a - E_\theta[\log[W(\mathbf{X})/\lambda(\theta)]] = a - \log a + C(\mathbf{X}, \theta),$$

where $C(\mathbf{X}, \theta) := -(1 + E_\theta[\log[W(\mathbf{X})/\lambda(\theta)]])$ does not depend on $a$. Differentiating $a - \log a$ with respect to $a$ and setting the result to zero yields $1 - 1/a = 0$ with unique solution $a = 1$. Since $1 - 1/a < 0$ for $a < 1$ and $1 - 1/a > 0$ for $a > 1$, we conclude that $a = 1$ in fact minimizes criterion (13). For the present exercise, we can then take $W(\mathbf{X}) := \hat{\tau}$ and $\lambda(\theta) := 1/\theta$ for data governed by a beta distribution as a special case.

6. Since $\hat{\tau}$ is an unbiased estimator of $\tau(\theta)$ whose variance tends to 0 as $n \to \infty$, we conclude by Technique #4 that $\hat{\tau}$ is consistent for $\tau(\theta)$.

7. Put $g(x) := 1/x$ for $x \in (0, \infty)$. We have $g''(x) = 2/x^3 > 0$, so that $g(x)$ is strictly convex. Noting that $\hat{\tau}$ is not degenerate, we then obtain

$$E_\theta[1/\hat{\tau}] = E_\theta[g(\hat{\tau})] > g(E_\theta[\hat{\tau}]) = 1/E_\theta[\hat{\tau}] = \theta$$

by Jensen's Inequality. (Technically we should also verify that $1/\hat{\tau}$ has finite expectation before invoking Jensen's Inequality, but that is not really necessary here: if $1/\hat{\tau}$ has infinite expectation, then obviously it is not unbiased for $\theta$.)

8. Put $\eta := 1/\theta$, $\hat{\eta} := \hat{\tau}$, and $\psi(x) := 1/x$ for $x \in (0, \infty)$. Since $\psi(x)$ is continuous, and since $\hat{\eta}$ is already known to be consistent for $\eta$, we apply Technique #1 to conclude that $\psi(\hat{\eta}) = 1/\hat{\tau}$ is consistent for $\psi(\eta) = \theta$.

9. Put $g(x) := -\log x$ for $x \in (0, 1)$. Then $g^{-1}(y) = \exp[-y]$ for $y \in (0, \infty)$ and the probability density function of $Y_1 := g(X_1)$ is

$$\theta(\exp[-y])^{\theta - 1} \left| \frac{d}{dy} \exp[-y] \right| = \theta \exp[-y\theta]$$

for $y \in (0, \infty)$. We recognize this as the probability density function of the exponential distribution with rate $\theta$ or mean $1/\theta$. As such, $S := \sum_{i=1}^n Y_i$ has the gamma distribution with shape $n$ and scale $1/\theta$. For $n \geq 2$ we have

$$E_\theta[1/S] = \int_0^\infty s^{-1} \frac{\theta^n}{\Gamma[n]} s^{n-1} \exp[-\theta s]\ ds = \frac{\theta^n \Gamma[n-1]}{\Gamma[n]\theta^{n-1}} = \frac{\theta}{n-1},$$

which yields

$$E_\theta[a/\hat{\tau}] = anE_\theta[1/S] = an\theta/(n-1).$$

Thus, $a/\hat{\tau}$ is unbiased for $\theta$ with $a = (n-1)/n$.

10. Since $1/\hat{\tau}$ is consistent for $\theta$ and $(n-1)/n$ converges in probability to 1, we apply Technique #2 to conclude that $(n-1)/(n\hat{\tau}) = a/\hat{\tau}$ is consistent for $\theta$.