BST 676 — Spring 2010 — Dr. Charnigo

Written Assignment 4 Solutions

1. The tables relevant to performing Fisher's exact test are of the form

	Medication	Placebo	Row Total
Adverse Event	x	$7-x$	7
No Adverse Event	$993-x$	$990+x$	1983
Column Total	993	997	1990

for $x \in \{0,1,2,3,4,5,6,7\}$.

The probabilities attached to these tables are 0.007839981, 0.05499062, 0.1649719, 0.2743993, 0.2732951, 0.1629883, 0.05389304, and 0.007621784 respectively. So, the p-value for Fisher's exact test of $H_0 : p_1 = p_2$ against $H_1 : p_1 \neq p_2$ is $0.05389304 + 0.007839981 + 0.007621784 = 0.0693548$ and we do not reject H_0. This agrees with the result I obtained from SAS, for which I used the code below.

```
data Trial;
input Treatment Adverse Count;
datalines;
0 0 6
1 0 1
0 1 987
1 1 996
run;
proc freq data=Trial;
weight Count;
tables Treatment*Adverse / chisq;
run;
```

2a. The likelihood function is

$$\prod_{i=1}^{n} (2\pi)^{-1/2} \exp[-(Y_i - \zeta_1 - \zeta_2 x_i)^2/2] = (2\pi)^{-n/2} \exp\left[-\sum_{i=1}^{n}(Y_i - \zeta_1 - \zeta_2 x_i)^2/2\right],$$

where ζ_1 denotes a putative value for α and ζ_2 a putative value for β.

2b. Maximizing

$$(2\pi)^{-n/2} \exp\left[-\sum_{i=1}^{n}(Y_i - \zeta_1 - \zeta_2 x_i)^2/2\right]$$

is the same as maximizing $\exp[-\sum_{i=1}^{n}(Y_i-\zeta_1-\zeta_2 x_i)^2/2]$, the same as maximizing $-\sum_{i=1}^{n}(Y_i-\zeta_1-\zeta_2 x_i)^2/2$, and hence the same as minimizing

$$\sum_{i=1}^{n}(Y_i - \zeta_1 - \zeta_2 x_i)^2.$$

Differentiating $\sum_{i=1}^{n}(Y_i - \zeta_1 - \zeta_2 x_i)^2$ in ζ_1 and setting the result to 0 yields, after multiplication by $-1/2$,

$$\sum_{i=1}^{n}(Y_i - \zeta_1 - \zeta_2 x_i) = \sum_{i=1}^{n} Y_i - \zeta_1 n - \zeta_2 \sum_{i=1}^{n} x_i = 0. \tag{1}$$

Likewise, differentiating $\sum_{i=1}^n (Y_i - \zeta_1 - \zeta_2 x_i)^2$ in ζ_2 yields

$$\sum_{i=1}^n (Y_i - \zeta_1 - \zeta_2 x_i) x_i = \sum_{i=1}^n x_i Y_i - \zeta_1 \sum_{i=1}^n x_i - \zeta_2 \sum_{i=1}^n x_i^2 = 0. \tag{2}$$

Collecting (1) and (2) into the matrix-vector equation

$$\begin{bmatrix} n & \sum_{i=1}^n x_i \\ \sum_{i=1}^n x_i & \sum_{i=1}^n x_i^2 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n Y_i \\ \sum_{i=1}^n x_i Y_i \end{bmatrix},$$

we obtain

$$\begin{aligned}
\begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} &= \begin{bmatrix} n & \sum_{i=1}^n x_i \\ \sum_{i=1}^n x_i & \sum_{i=1}^n x_i^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum_{i=1}^n Y_i \\ \sum_{i=1}^n x_i Y_i \end{bmatrix} \\
&= \begin{bmatrix} \sum_{i=1}^n x_i^2 & -\sum_{i=1}^n x_i \\ -\sum_{i=1}^n x_i & n \end{bmatrix} \begin{bmatrix} \sum_{i=1}^n Y_i \\ \sum_{i=1}^n x_i Y_i \end{bmatrix} \Bigg/ \left(n \sum_{i=1}^n x_i^2 - \left\{ \sum_{i=1}^n x_i \right\}^2 \right) \\
&= \begin{bmatrix} (\sum_{i=1}^n x_i^2 \sum_{i=1}^n Y_i - \sum_{i=1}^n x_i \sum_{i=1}^n x_i Y_i) \\ (-\sum_{i=1}^n x_i \sum_{i=1}^n Y_i + n \sum_{i=1}^n x_i Y_i) \end{bmatrix} \Bigg/ \left(n \sum_{i=1}^n x_i^2 - \left\{ \sum_{i=1}^n x_i \right\}^2 \right) \\
&=: \begin{bmatrix} \widehat{\alpha} \\ \widehat{\beta} \end{bmatrix}.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\widehat{\alpha} &= \left(\sum_{i=1}^n x_i^2 \sum_{i=1}^n Y_i - \sum_{i=1}^n x_i \sum_{i=1}^n x_i Y_i \right) \Bigg/ \left(n \sum_{i=1}^n x_i^2 - \left\{ \sum_{i=1}^n x_i \right\}^2 \right) \\
&= \left(\sum_{i=1}^n x_i^2 \sum_{i=1}^n Y_i - \left\{ \sum_{i=1}^n x_i \right\}^2 \bar{Y} + \left\{ \sum_{i=1}^n x_i \right\}^2 \bar{Y} - \sum_{i=1}^n x_i \sum_{i=1}^n x_i Y_i \right) \Bigg/ \left(n \sum_{i=1}^n x_i^2 - \left\{ \sum_{i=1}^n x_i \right\}^2 \right) \\
&= \left(n \sum_{i=1}^n x_i^2 \bar{Y} - \left\{ \sum_{i=1}^n x_i \right\}^2 \bar{Y} - n\bar{x} \sum_{i=1}^n x_i Y_i + \bar{x} \sum_{i=1}^n x_i \sum_{i=1}^n Y_i \right) \Bigg/ \left(n \sum_{i=1}^n x_i^2 - \left\{ \sum_{i=1}^n x_i \right\}^2 \right) \\
&= \bar{Y} - \widehat{\beta} \bar{x}.
\end{aligned} \tag{3}$$

2c. Differentiating $\sum_{i=1}^n (Y_i - \zeta_1)^2$ in ζ_1 and setting the result to 0 yields, after multiplication by $-1/2$,

$$\sum_{i=1}^n (Y_i - \zeta_1) = \sum_{i=1}^n Y_i - n\zeta_1 = 0,$$

whose solution is

$$\zeta_1 = \bar{Y} =: \widehat{\alpha}_0.$$

2d. Put $\widehat{Y}_i := \widehat{\alpha} + \widehat{\beta} x_i$. We have

$$\lambda = \frac{(2\pi)^{-n/2} \exp[-\sum_{i=1}^n (Y_i - \bar{Y})^2/2]}{(2\pi)^{-n/2} \exp[-\sum_{i=1}^n (Y_i - \widehat{Y}_i)^2/2]} = \exp\left[-\sum_{i=1}^n (Y_i - \bar{Y})^2/2 + \sum_{i=1}^n (Y_i - \widehat{Y}_i)^2/2 \right],$$

so that

$$-2 \log \lambda = \sum_{i=1}^n (Y_i - \bar{Y})^2 - \sum_{i=1}^n (Y_i - \widehat{Y}_i)^2.$$

From the well-known relation

$$\text{Reg SS} \ + \ \text{Res SS} \ = \ \text{Tot SS}, \tag{4}$$

we conclude that

$$-2\log\lambda = \sum_{i=1}^{n}(\widehat{Y}_i - \bar{Y})^2.$$

As an aside, here is a proof of (4). We have

$$\begin{aligned}
\sum_{i=1}^{n}(Y_i - \bar{Y})^2 &= \sum_{i=1}^{n}(Y_i - \widehat{Y}_i + \widehat{Y}_i - \bar{Y})^2 \\
&= \sum_{i=1}^{n}(Y_i - \widehat{Y}_i)^2 + \sum_{i=1}^{n}(\widehat{Y}_i - \bar{Y})^2 + 2\sum_{i=1}^{n}(Y_i - \widehat{Y}_i)(\widehat{Y}_i - \bar{Y}),
\end{aligned}$$

so verifying that

$$\sum_{i=1}^{n}(Y_i - \widehat{Y}_i)(\widehat{Y}_i - \bar{Y}) = 0 \tag{5}$$

is sufficient. We have

$$\begin{aligned}
\sum_{i=1}^{n}(Y_i - \widehat{Y}_i)(\widehat{Y}_i - \bar{Y}) &= \sum_{i=1}^{n}(Y_i - \widehat{\alpha} - \widehat{\beta}x_i)(\widehat{\alpha} + \widehat{\beta}x_i - \bar{Y}) \\
&= (\widehat{\alpha} - \bar{Y})\sum_{i=1}^{n}(Y_i - \widehat{\alpha} - \widehat{\beta}x_i) \qquad (6) \\
&\quad + \widehat{\beta}\sum_{i=1}^{n}x_i(Y_i - \widehat{\alpha} - \widehat{\beta}x_i). \qquad (7)
\end{aligned}$$

Line (6) is zero because of (1), while line (7) is zero because of (2).

2e. Rearranging our earlier formula for $\widehat{\beta}$ yields

$$\widehat{\beta} = \sum_{i=1}^{n}(x_i - \bar{x})Y_i \Big/ \sum_{i=1}^{n}(x_i - \bar{x})^2,$$

which demonstrates the normality of $\widehat{\beta}$ since each Y_i is normally distributed and a linear combination of independent normal random variables is itself normally distributed. Moreover, by the linearity properties of expectation and variance we have

$$\begin{aligned}
E[\widehat{\beta}] &= \sum_{i=1}^{n}(x_i - \bar{x})(\alpha + \beta x_i) \Big/ \sum_{i=1}^{n}(x_i - \bar{x})^2 \\
&= \alpha \times \sum_{i=1}^{n}(x_i - \bar{x}) \Big/ \sum_{i=1}^{n}(x_i - \bar{x})^2 + \beta \times \sum_{i=1}^{n}x_i(x_i - \bar{x}) \Big/ \sum_{i=1}^{n}(x_i - \bar{x})^2 \\
&= \alpha \times 0 + \beta \times \sum_{i=1}^{n}(x_i - \bar{x})^2 \Big/ \sum_{i=1}^{n}(x_i - \bar{x})^2 \\
&= \beta
\end{aligned}$$

and

$$Var[\widehat{\beta}] = \sum_{i=1}^{n}(x_i - \bar{x})^2 \Big/ \left(\sum_{i=1}^{n}(x_i - \bar{x})^2\right)^2 = 1 \Big/ \sum_{i=1}^{n}(x_i - \bar{x})^2.$$

2f. We have

$$\sum_{i=1}^{n}(\widehat{Y}_i-\bar{Y})^2=\sum_{i=1}^{n}(\widehat{\alpha}+\widehat{\beta}x_i-\bar{Y})^2=\sum_{i=1}^{n}(\bar{Y}-\widehat{\beta}\bar{x}+\widehat{\beta}x_i-\bar{Y})^2=\widehat{\beta}^2\sum_{i=1}^{n}(x_i-\bar{x})^2,$$

which has the chi-square distribution on one degree of freedom under H_0. Rejecting H_0 if $-2\log\lambda>\chi^2_{1,1-\alpha}$ yields a test with significance level α, so the critical value for λ is $\exp[-\chi^2_{1,1-\alpha}/2]$. Here $\chi^2_{1,1-\alpha}$ denotes the upper α quantile of the chi-square distribution on one degree of freedom.

3a. Put $\widehat{\theta}:=\bar{X}$. Then $\widehat{\theta}$ is unbiased with variance θ/n and asymptotically normal (by the Central Limit Theorem), so that

$$\frac{\widehat{\theta}-\theta}{\sqrt{\theta/n}}\xrightarrow{L}N(0,1).$$

Since $\widehat{\theta}$ is consistent for θ (by the Weak Law of Large Numbers), the Continuous Mapping Theorem yields

$$\frac{\sqrt{\theta/n}}{\sqrt{\widehat{\theta}/n}}=\frac{\sqrt{\theta}}{\sqrt{\widehat{\theta}}}\xrightarrow{P}1,$$

whence Slutsky's Theorem #3 provides

$$\frac{\widehat{\theta}-\theta}{\sqrt{\widehat{\theta}/n}}\xrightarrow{L}N(0,1).$$

If H_0 is true, then

$$\frac{\widehat{\theta}-\theta_0}{\sqrt{\widehat{\theta}/n}}\xrightarrow{L}N(0,1),\tag{8}$$

so that a Wald test may be defined by rejecting H_0 if the left member of (8) exceeds $z_{1-\alpha/2}$ in absolute value.

3b. The score statistic is

$$S(\theta;\mathbf{X}):=\sum_{i=1}^{n}\frac{\partial}{\partial\theta}\log f(X_i;\theta)=-n+\sum_{i=1}^{n}X_i/\theta,$$

whose variance is $J_n(\theta)=n/\theta$. If H_0 is true, then

$$\frac{\sum_{i=1}^{n}X_i/\theta_0-n}{\sqrt{n/\theta_0}}=\frac{\widehat{\theta}-\theta_0}{\sqrt{\theta_0/n}}\xrightarrow{L}N(0,1),\tag{9}$$

so that a score test may be defined by rejecting H_0 if the left member of (9) exceeds $z_{1-\alpha/2}$ in absolute value.