BST 676 – Spring 2011 – Dr. Charnigo

Unit VII: Evaluating Interval Estimators

a. Motivating case study #1: When should one-sided intervals be used?

We saw in Unit VI that inverting a one-sided test could yield a one-sided confidence interval, in the sense that either the lower bound for the parameter would be the infimum of the parameter space ($-\infty$ if $\Theta = \mathbb{R}$) or the upper bound for the parameter would be the supremum of the parameter space ($+\infty$ if $\Theta = \mathbb{R}$).

Now we pose the question of why we might wish to use one-sided confidence intervals.

A theoretical reason for considering one-sided confidence intervals requires material to be discussed in Unit VII, so this motivating case study will not be fully resolved at this time. However, we can still provide some practical reasons for considering one-sided confidence intervals.

To make the discussion concrete, suppose that we are interested in estimating a risk difference $p_1 - p_2$ and that we wish to employ a Wald interval.

A two-sided Wald interval is

$$[\hat{p}_1 - \hat{p}_2 - 1.96\sqrt{\hat{p}_1(1-\hat{p}_1)/n_1 + \hat{p}_2(1-\hat{p}_2)/n_2},$$

$$\hat{p}_1 - \hat{p}_2 + 1.96\sqrt{\hat{p}_1(1-\hat{p}_1)/n_1 + \hat{p}_2(1-\hat{p}_2)/n_2}].$$

A one-sided Wald interval is

$$[-1\ ,\ \hat{p}_1 - \hat{p}_2 + 1.645\sqrt{\hat{p}_1(1-\hat{p}_1)/n_1 + \hat{p}_2(1-\hat{p}_2)/n_2}] \tag{1}$$

while another one-sided Wald interval is

$$[\hat{p}_1 - \hat{p}_2 - 1.645\sqrt{\hat{p}_1(1-\hat{p}_1)/n_1 + \hat{p}_2(1-\hat{p}_2)/n_2}\ ,\ 1]. \tag{2}$$

Suppose further that p_1 represents the risk on a new treatment while p_2 represents the risk on an existing treatment. If our only concern is in determining whether $p_1 - p_2$ is *less* than some ambivalence point, which can be accomplished by comparing an upper bound for $p_1 - p_2$ to the ambivalence point, then why estimate a lower bound for $p_1 - p_2$?

Indeed, interval (1) enjoys an advantage over the two-sided interval in that the upper bound exceeds the point estimator by 1.645 times the standard error rather than by 1.96 times the standard error. Hence, we are more likely to conclude that $p_1 - p_2$ is less than the ambivalence point, if in fact this is true, using interval (1) than using the two-sided interval.

An analogy may be drawn to power for one-sided tests. Generally speaking, a one-sided test is more powerful than the companion two-sided test, provided that you have guessed correctly about the direction of departure from the null hypothesis.

While superiority studies rarely employ one-sided tests or one-sided confidence intervals, some noninferiority studies do use one-sided confidence intervals of the form (1). Recall from our discussion of the second motivating case study in Unit V that noninferiority studies are distinguished from superiority studies in that the former are characterized by positive ambivalence points while the latter are characterized by negative (or zero) ambivalence points.

A one-sided confidence interval of the form (2) is more difficult to envisage in practice, but one possibility emerges if p_1 represents not the risk on a new treatment but rather the probability of success on a new treatment and similarly p_2 represents the probability of success on an existing treatment. Then our interest may lie in determining whether $p_1 - p_2$ is *greater* than some ambivalence point, which can be accomplished by comparing a lower bound for $p_1 - p_2$ to the ambivalence point.

We will return to this motivating case study at the end of Unit VII with a theoretical consideration.

b. Motivating case study #2: How "approximate" is an approximate 95% interval?

We saw in Unit VI that a confidence interval formula intended for normal data could fare quite poorly for non-normal data if the sample size were small. While the performance of a confidence interval formula can always be investigated with a simulation experiment, as was done in Unit VI, there are some situations in which the performance can also be determined analytically.

For instance, let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \theta^{-1}\exp[-x\theta^{-1}]$ for $x \in (0, \infty)$ and $\theta \in (0, \infty)$. Let $G(t)$ denote the cumulative distribution function for the gamma distribution with shape n and scale 1. One can show that, for any constant c, we have

$$P(\bar{X} - c\bar{X}/\sqrt{n} < \theta < \bar{X} + c\bar{X}/\sqrt{n}) = G\left(\frac{n}{1 - c/\sqrt{n}}\right) - G\left(\frac{n}{1 + c/\sqrt{n}}\right). \quad (3)$$

The right member of (3) can, of course, be evaluated easily in R given choices of n and c. Also, note that the right member of (3) is neither an estimate nor an approximation.

Actually, we already have the probabilistic machinery to resolve this motivating case study. However, we may not yet have sufficient insight on how to use the probabilistic machinery. Therefore, we will return to this motivating case study at the end of Unit VII, after discussing a strategy that sometimes works for the analytic determination of how well a confidence interval formula performs.

c. Controlling the length of an interval

The length of a two-sided confidence interval is generally influenced by the sample size and, in many commonly encountered situations, is roughly proportional to $1/\sqrt{n}$.

When a two-sided confidence interval has been obtained by pivoting, another factor influencing its length is our selection of quantiles from the distribution of the pivot.

For example, suppose that $X_1, X_2, \ldots, X_n \overset{iid}{\sim} N(\theta, 1)$ for $\theta \in \mathbb{R}$. If we use $\sqrt{n}(\bar{X} - \theta)$ as a pivot, then

$$a < \sqrt{n}(\bar{X} - \theta) < b$$

results in the interval

$$\bar{X} - b/\sqrt{n} < \theta < \bar{X} - a/\sqrt{n}. \quad (4)$$

The length of (4) is $(b - a)/\sqrt{n}$.

Let $g(t)$ denote the probability density function of the pivot. In this example, we have

$$g(t) = (2\pi)^{-1/2} \exp[-t^2/2].$$

If we wish to obtain a $100(1-\alpha)\%$ confidence interval, then we must impose the constraint

$$\int_a^b g(t)\;dt = 1-\alpha, \tag{5}$$

which renders b a function of a. Differentiating both sides of (5) with respect to a yields

$$g(b)\frac{db}{da} - g(a) = 0,$$

whence

$$\frac{db}{da} = \frac{g(a)}{g(b)}. \tag{6}$$

Assuming that n is fixed, we can minimize the length of the confidence interval by minimizing $b-a$. Recalling that b is a function of a, we differentiate $b-a$ with respect to a and set the result equal to zero,

$$\frac{db}{da} - 1 = 0. \tag{7}$$

Substituting (6) into (7) yields

$$g(a) = g(b),$$

which in this example simplifies to

$$a^2 = b^2. \tag{8}$$

Since $a = b$ cannot satisfy (5), we conclude that $a = -b < 0 < b$. Moreover, (5) is satisfied by choosing $a = z_{\alpha/2} = -z_{1-\alpha/2} = -b$.

Consider a second example. Let $X_1, X_2, \ldots, X_n \stackrel{iid}{\sim} 1_{\{x>0\}}\theta^{-1}\exp[-x\theta^{-1}]$ for $\theta \in (0,\infty)$. Employing $(X_1 + X_2 + \cdots + X_n)/\theta$ as a pivot, we rearrange

$$a < (X_1 + X_2 + \cdots + X_n)/\theta < b$$

as

$$(X_1 + X_2 + \cdots + X_n)/b < \theta < (X_1 + X_2 + \cdots + X_n)/a. \tag{9}$$

The length of (9) is random but proportional to $(1/a - 1/b)$.

Let $g(t)$ denote the probability density function of the pivot. In this example, we have

$$g(t) = 1_{\{t>0\}}t^{n-1}\exp[-t]/\Gamma[n].$$

As in the first example, we have

$$\int_a^b g(t)\ dt = 1 - \alpha \tag{10}$$

and

$$\frac{db}{da} = \frac{g(a)}{g(b)}. \tag{11}$$

Minimizing $(1/a - 1/b)$ produces

$$-\frac{1}{a^2} + \frac{db}{da} \times \frac{1}{b^2} = 0. \tag{12}$$

Substitution of (12) into (11) yields

$$g(b)b^2 = g(a)a^2,$$

which in this example simplifies to

$$b^{n+1} \exp[-b] = a^{n+1} \exp[-a]. \tag{13}$$

The simultaneous solution to (10) and (13) may be found through the reparametrization $a = g_{n,1,c_1}$ and $b = g_{n,1,c_1+1-\alpha}$ and a search over $c_1 \in (0, \alpha)$ for the solution to

$$g_{n,1,c_1+1-\alpha}^{n+1} \exp[-g_{n,1,c_1+1-\alpha}] = g_{n,1,c_1}^{n+1} \exp[-g_{n,1,c_1}]. \tag{14}$$

When $n = 4$, the solution to (14) is $c_1 = 0.0484$, confirming the answer obtained in Unit VI.

To summarize, our general strategy for minimizing the length of a two-sided confidence interval obtained through pivoting is as follows. If the length is proportional to $h(b) - h(a)$, then we must simultaneously solve

$$\int_a^b g(t)\ dt = 1 - \alpha$$

and

$$g(b)/h'(b) = g(a)/h'(a).$$

This can be accomplished by reparametrizing $a = q_{c_1}$ and $b = q_{c_1+1-\alpha}$, where q_c denotes the c quantile of $g(t)$, and searching over $c_1 \in (0, \alpha)$ for the solution to

$$g(q_{c_1+1-\alpha})/h'(q_{c_1+1-\alpha}) = g(q_{c_1})/h'(q_{c_1}).$$

d. Unbiased and uniformly most accurate intervals

Let $X_1, X_2, \ldots, X_n \stackrel{iid}{\sim} f(x;\theta)$, where θ is an unknown parameter belonging to a known parameter space Θ. Suppose that $[L(\mathbf{X}), U(\mathbf{X})]$ is a confidence interval for θ such that

$$P_\theta(\theta \in [L(\mathbf{X}), U(\mathbf{X})]) \geq P_\theta(\zeta \in [L(\mathbf{X}), U(\mathbf{X})])$$

for any $\zeta \in \Theta$. Then we say that $[L(\mathbf{X}), U(\mathbf{X})]$ is unbiased. In words, an unbiased confidence interval is more likely to contain the true parameter θ than any other element in the parameter space. Clearly, unbiasedness is a desirable property for a confidence interval.

Typically, we obtain unbiased confidence intervals by inverting unbiased tests. For instance, suppose that

$$f(x;\theta) = (2\pi)^{-1/2} \exp[-(x-\theta)^2/2]$$

with $\Theta = \mathbb{R}$ and that we are testing $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$. We already know from Example #1 of Unit V that rejecting the null hypothesis whenever $\sqrt{n}|\bar{X} - \theta_0| > z_{1-\alpha/2}$ produces an unbiased test of level α.

Moreover, we know from our undergraduate methods course that inverting this test yields the $100(1-\alpha)\%$ confidence interval

$$[\bar{X} - z_{1-\alpha/2}/\sqrt{n}, \bar{X} + z_{1-\alpha/2}/\sqrt{n}].$$

We have

$$\begin{aligned}
& P_\theta(\zeta \in [\bar{X} - z_{1-\alpha/2}/\sqrt{n}, \bar{X} + z_{1-\alpha/2}/\sqrt{n}]) && (15)\\
= \;& P_\theta(\bar{X} - z_{1-\alpha/2}/\sqrt{n} \leq \zeta \leq \bar{X} + z_{1-\alpha/2}/\sqrt{n}) \\
= \;& P_\theta(\zeta - z_{1-\alpha/2}/\sqrt{n} \leq \bar{X} \leq \zeta + z_{1-\alpha/2}/\sqrt{n}) \\
= \;& P_\theta(\zeta - z_{1-\alpha/2}/\sqrt{n} - \theta \leq \bar{X} - \theta \leq \zeta + z_{1-\alpha/2}/\sqrt{n} - \theta) \\
= \;& P_\theta(\sqrt{n}(\zeta - \theta) - z_{1-\alpha/2} \leq \sqrt{n}(\bar{X} - \theta) \leq \sqrt{n}(\zeta - \theta) + z_{1-\alpha/2}) \\
= \;& \Phi[\sqrt{n}(\zeta - \theta) + z_{1-\alpha/2}] - \Phi[\sqrt{n}(\zeta - \theta) - z_{1-\alpha/2}]. && (16)
\end{aligned}$$

Let us maximize (16) over $\zeta \in \Theta$. Differentiating in ζ, we obtain

$$\sqrt{n}\phi[\sqrt{n}(\zeta - \theta) + z_{1-\alpha/2}] - \sqrt{n}\phi[\sqrt{n}(\zeta - \theta) - z_{1-\alpha/2}],$$

which equals zero if and only if

$$\sqrt{n}(\zeta - \theta) + z_{1-\alpha/2} = \pm\{\sqrt{n}(\zeta - \theta) - z_{1-\alpha/2}\}. \tag{17}$$

The "+" contingency in (17) is impossible, while the "−" contingency occurs if and only if $\zeta = \theta$. Thus, (15) is maximized when $\zeta = \theta$.

If you need to be convinced that we have a maximum rather than a minimum, then let ζ pass to $\pm\infty$ in (16).

In conclusion, the confidence interval

$$[\bar{X} - z_{1-\alpha/2}/\sqrt{n}, \bar{X} + z_{1-\alpha/2}/\sqrt{n}]$$

is unbiased.

Another desirable property for a confidence interval is that of being uniformly most accurate. However, this property is typically encountered only in one-sided confidence intervals, since its occurrence is linked to the inversion of a uniformly most powerful test.

We say that a $100(1-\alpha)\%$ confidence interval $[L^*(\mathbf{X}), \infty)$ is uniformly most accurate if

$$P_\theta(\zeta \in [L^*(\mathbf{X}), \infty)) \leq P_\theta(\zeta \in [L(\mathbf{X}), \infty)), \tag{18}$$

where $[L(\mathbf{X}), \infty)$ is any other $100(1-\alpha)\%$ confidence interval and $\zeta < \theta$. In words, a uniformly most accurate confidence interval $[L^*(\mathbf{X}), \infty)$ minimizes the probability of containing a false element of the parameter space if that false element is less than the true parameter.

We say that a $100(1-\alpha)\%$ confidence interval $(-\infty, U^*(\mathbf{X})]$ is uniformly most accurate if

$$P_\theta(\zeta \in (-\infty, U^*(\mathbf{X})]) \leq P_\theta(\zeta \in (-\infty, U(\mathbf{X})]), \tag{19}$$

where $(-\infty, U(\mathbf{X})]$ is any other $100(1-\alpha)\%$ confidence interval and $\zeta > \theta$. In words, a uniformly most accurate confidence interval $(-\infty, U^*(\mathbf{X})]$ minimizes the probability of containing a false element of the parameter space if that false element is greater than the true parameter.

An obvious question is why require (18) only for $\zeta < \theta$ rather than all $\zeta \in \Theta$ and (19) only for $\zeta > \theta$ rather than all $\zeta \in \Theta$. The answer is also obvious, after a moment's reflection.

Suppose that $\theta \in [L^*(\mathbf{X}), \infty)$, which we will all agree is desirable. Then, necessarily, $\zeta \in [L^*(\mathbf{X}), \infty)$ for any $\zeta > \theta$. Therefore, expecting $[L^*(\mathbf{X}), \infty)$ to exclude $\zeta > \theta$ is unreasonable.

Likewise, expecting $(-\infty, U^*(\mathbf{X})]$ to exclude $\zeta < \theta$ is unreasonable.

Conditions (18) and (19) cannot be verified directly, since in general there will be infinitely many competitors of the form $[L(\mathbf{X}), \infty)$ and $(-\infty, U(\mathbf{X})]$. However, as stated earlier, we obtain a uniformly most accurate confidence interval by inverting a uniformly most powerful test.

So, for instance, suppose that

$$f(x;\theta) = 1_{\{x>0\}}\theta \exp[-x\theta]$$

with $\Theta = (0, \infty)$ and that we are testing $H_0 : \theta \leq \theta_0$ against $H_1 : \theta > \theta_0$.

We saw in Unit V that a uniformly most powerful test of level α was obtained by rejecting the null hypothesis if $\theta_0 \sum_{i=1}^n X_i < g_{n,1,\alpha}$, the α quantile of the gamma distribution with shape n and scale 1. Thus, a uniformly most powerful test of level α entails accepting the null hypothesis if $\theta_0 \sum_{i=1}^n X_i \geq g_{n,1,\alpha}$.

Since $\theta_0 \sum_{i=1}^n X_i \geq g_{n,1,\alpha}$ is equivalent to $\theta_0 \geq g_{n,1,\alpha}/\sum_{i=1}^n X_i$, inversion yields the confidence interval

$$\left[g_{n,1,\alpha}/\sum_{i=1}^n X_i, \infty\right).$$

This is a uniformly most accurate $100(1-\alpha)\%$ confidence interval for θ among all $100(1-\alpha)\%$ confidence intervals of the form $[L(\mathbf{X}), \infty)$.

Perhaps you can intuit that

$$\left(0, g_{n,1,1-\alpha}/\sum_{i=1}^n X_i\right] \tag{20}$$

is a uniformly most accurate $100(1-\alpha)\%$ confidence interval for θ among all $100(1-\alpha)\%$ confidence intervals of the form $(0, U(\mathbf{X})]$. If your intuition does not suffice, you can verify my claim by finding the uniformly most powerful test of $H_0 : \theta \geq \theta_0$ against $H_1 : \theta < \theta_0$ and then inverting.

Finally, note that I have harmlessly replaced $-\infty$ by 0 in (20) since Θ is bounded below by 0.

e. Comparing nominal and actual confidence levels

Now we provide a strategy for the analytic determination of how well a confidence interval formula performs in situations when the confidence interval is given by

$$k_1 T \pm k_2 \sqrt{k_3 + k_4 T + k_5 T^2}, \tag{21}$$

where k_1, k_2, k_3, k_4, k_5 are constants and T is function of $X_1, X_2, \ldots, X_n$ whose probabilistic behavior is known.

For example, suppose that

$$X_1, X_2, \ldots, X_n \overset{iid}{\sim} \theta^x (1-\theta)^{1-x}$$

for $x \in \{0, 1\}$ and $\theta \in (0, 1)$. The Wald interval with 95% nominal confidence level has the form (21) with

$$T := \sum_{i=1}^{n} X_i \sim Binom(\theta, n),$$

$k_1 := 1/n$, $k_2 := 1.96$, $k_3 := 0$, $k_4 := 1/n^2$, and $k_5 := -1/n^3$.

Our strategy begins by noting the algebraic equivalence between

$$\theta \in [k_1 T - k_2 \sqrt{k_3 + k_4 T + k_5 T^2}\ ,\ k_1 T + k_2 \sqrt{k_3 + k_4 T + k_5 T^2}]$$

and

$$(k_1 T - \theta)^2 / k_2^2 = k_1^2 T^2 / k_2^2 - 2 k_1 T \theta / k_2^2 + \theta^2 / k_2^2 \leq k_3 + k_4 T + k_5 T^2,$$

which may be rearranged as

$$(k_1^2/k_2^2 - k_5) T^2 - (2 k_1 \theta / k_2^2 + k_4) T + (\theta^2 / k_2^2 - k_3) \leq 0. \tag{22}$$

We anticipate that (22) will be satisfied when

$$T \geq \frac{(2k_1\theta/k_2^2 + k_4) - \sqrt{(2k_1\theta/k_2^2 + k_4)^2 - 4(k_1^2/k_2^2 - k_5)(\theta^2/k_2^2 - k_3)}}{2(k_1^2/k_2^2 - k_5)} \tag{23}$$

and

$$T \leq \frac{(2k_1\theta/k_2^2 + k_4) + \sqrt{(2k_1\theta/k_2^2 + k_4)^2 - 4(k_1^2/k_2^2 - k_5)(\theta^2/k_2^2 - k_3)}}{2(k_1^2/k_2^2 - k_5)}. \tag{24}$$

We can then evaluate the probability that (23) and (24) are satisfied.

Continuing our example with Bernoulli data, expressions (23) and (24) become

$$T \geq \frac{2\theta/(3.84n) + 1/n^2 - \sqrt{(2\theta/(3.84n) + 1/n^2)^2 - 4(1/(3.84n^2) + 1/n^3)\theta^2/3.84}}{(2/(3.84n^2) + 2/n^3)} \tag{25}$$

and

$$T \leq \frac{2\theta/(3.84n) + 1/n^2 + \sqrt{(2\theta/(3.84n) + 1/n^2)^2 - 4(1/(3.84n^2) + 1/n^3)\theta^2/3.84}}{(2/(3.84n^2) + 2/n^3)}, \tag{26}$$

respectively.

If $\theta = 0.5$ and $n = 30$, then expressions (25) and (26) evaluate to 9.95 and 20.05. In this case, the actual confidence level of the Wald interval is seen to be

$$\sum_{k=10}^{20} \binom{30}{k} 0.5^k (1 - 0.5)^{30-k} = 0.957226. \tag{27}$$

As a check of our analytic determination, we can perform the simulation experiment for which R code is shown below. Five runs of the simulation experiment yielded the following estimates of the actual confidence level: 0.9580, 0.9569, 0.9581, 0.9568, and 0.9566. All five estimates are close to 0.957226.

```
data<-rbinom(10000,size=30,prob=.5)
hatp<-data/30
lower<- hatp-qnorm(.975)*sqrt(hatp*(1-hatp)/30)
upper<- hatp+qnorm(.975)*sqrt(hatp*(1-hatp)/30)
mean( (lower < 0.5) & (upper > 0.5) )
```

A final point is that the actual confidence level exceeding the nominal confidence level as in (27) is not a universal phenomenon for the Wald interval with Bernoulli data. For instance, if $\theta = 0.5$ and $n = 34$, then the actual confidence level of the Wald interval is 0.9423873. The actual confidence level may fall on either side of the nominal confidence level because of the discreteness of the data and, in particular, the fact that (25) and (26) may fall just above or just below integer values.

f. Resolution of motivating case studies

Our first motivating case study sought a rationale for using one-sided confidence intervals. From a practical perspective, one-sided confidence intervals of the form $(-\infty, U(\mathbf{X})]$ are more likely to admit an inference that a parameter of interest (such as a risk difference) lies below an ambivalence point than are two-sided confidence intervals. This advantage is sufficiently compelling that one-sided confidence intervals are often used in noninferiority studies.

We have also seen that one-sided confidence intervals have a theoretical advantage, namely that they can possess the property of being uniformly most accurate. For instance, a uniformly most accurate confidence interval of the form $[L(\mathbf{X}), \infty)$ has the smallest possible probability of including ζ, where ζ is an element of the parameter space less than θ.

As another application of one-sided confidence intervals, suppose that we want to estimate a lower bound for the 2010 prevalence of some disease based on data

$$X_1, X_2, \ldots, X_n \overset{iid}{\sim} 1_{\{x \in \{0,1\}\}} \theta^x (1-\theta)^{1-x}$$

for some $\theta \in (0, 1)$, where $X_i = 1$ if person i has the disease.

One can show that a uniformly most powerful test of $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$ entails rejecting the null hypothesis if $\sum_{i=1}^n X_i > t(\theta_0)$, the cutoff $t(\theta_0)$ an increasing function of θ_0 chosen to yield a desired significance level. Thus, a uniformly most accurate confidence interval contains any θ_0 such that $t(\theta_0) \geq \sum_{i=1}^n X_i$. Note that we are, in essence, trying to minimize the probability of understating the disease burden.

Of course, the above formulation is rather opaque, but application of the Central Limit Theorem yields the tractable approximation

$$t(\theta_0) \approx z_{1-\alpha}\sqrt{n\theta_0(1-\theta_0)} + n\theta_0,$$

from which an approximate lower bound for θ can be determined by taking the smaller root of the quadratic (in θ_0) equation

$$\left(\sum_{i=1}^n X_i - n\theta_0\right)^2 = \chi^2_{1,1-2\alpha} n\theta_0(1-\theta_0).$$

Our second motivating case study sought to establish

$$P(\bar{X} - c\bar{X}/\sqrt{n} < \theta < \bar{X} + c\bar{X}/\sqrt{n}) = G\left(\frac{n}{1 - c/\sqrt{n}}\right) - G\left(\frac{n}{1 + c/\sqrt{n}}\right), \quad (28)$$

where $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \theta^{-1}\exp[-x\theta^{-1}]$ for $x \in (0, \infty)$, $\theta \in (0, \infty)$, and $G(t)$ the cumulative distribution function for the gamma distribution with shape n and scale 1.

Noting that the proposed interval has the form (21) with

$$T := \sum_{i=1}^{n} X_i/\theta \sim G(t),$$

$k_1 := \theta/n$, $k_2 := c$, $k_3 := 0$, $k_4 := 0$, and $k_5 := \theta^2/n^3$, we seek the probability that

$$T \geq \frac{2\theta^2/(nc^2) - \sqrt{(2\theta^2/(nc^2))^2 - 4(\theta^2/(n^2c^2) - \theta^2/n^3)(\theta^2/c^2)}}{2(\theta^2/(n^2c^2) - \theta^2/n^3)} \quad (29)$$

and

$$T \leq \frac{2\theta^2/(nc^2) + \sqrt{(2\theta^2/(nc^2))^2 - 4(\theta^2/(n^2c^2) - \theta^2/n^3)(\theta^2/c^2)}}{2(\theta^2/(n^2c^2) - \theta^2/n^3)}. \quad (30)$$

Factoring out $2\theta^2$ from the numerators and denominators of (29) and (30), and then multiplying by nc^2, we obtain

$$T \geq \frac{1 - \sqrt{c^2/n}}{n^{-1}(1 - c^2/n)} = \frac{1 - c^2/n}{n^{-1}(1 - c^2/n)(1 + \sqrt{c^2/n})} = \frac{n}{1 + c/\sqrt{n}} \quad (31)$$

and

$$T \leq \frac{1 + \sqrt{c^2/n}}{n^{-1}(1 - c^2/n)} = \frac{1 - c^2/n}{n^{-1}(1 - c^2/n)(1 - \sqrt{c^2/n})} = \frac{n}{1 - c/\sqrt{n}}. \quad (32)$$

The probability that requirements (31) and (32) are satisfied is

$$P\left(\frac{n}{1 + c/\sqrt{n}} \leq T \leq \frac{n}{1 - c/\sqrt{n}}\right) = G\left(\frac{n}{1 - c/\sqrt{n}}\right) - G\left(\frac{n}{1 + c/\sqrt{n}}\right),$$

which validates (28).