BST 676 — Spring 2011 — Dr. Charnigo

Written Assignment 1 Solutions

1a. Since $E[X_1] = Var[X_1] = \theta$, the Central Limit Theorem tells us that $\sqrt{n}(\bar{X}_n - \theta) \xrightarrow{L} N(0, \theta)$.

1b. We have $\bar{X}_n \xrightarrow{P} \theta$ by the Weak Law of Large Numbers. By the Continuous Mapping Theorem with $h(y) := y^{-1/2} 1_{\{y \in (0,\infty)\}}$ we have $h(\bar{X}_n) \xrightarrow{P} h(\theta) = \theta^{-1/2}$. Since $h(\bar{X}_n) = \bar{X}_n^{-1/2}$ with probability approaching 1 as $n \to \infty$, we also have $\bar{X}_n^{-1/2} \xrightarrow{L} \theta^{-1/2}$. Slutsky's Theorem #3 then yields $\bar{X}_n^{-1/2} \times \sqrt{n}(\bar{X}_n - \theta) \xrightarrow{L} \theta^{-1/2} \times N(0, \theta) = N(0, 1)$.

1c. We have $P[-1.96 \le \bar{X}_n^{-1/2} \sqrt{n}(\bar{X}_n - \theta) \le 1.96] \approx 0.95$ for large n. Algebraic rearrangement of the compound inequality inside the probability operator yields the approximate 95% confidence interval $\bar{X}_n - 1.96\sqrt{\bar{X}_n/n} \le \theta \le \bar{X}_n + 1.96\sqrt{\bar{X}_n/n}$.

1d. Assuming that g has a continuous derivative in a neighborhood of θ, the delta method yields $\sqrt{n}(g(\bar{X}_n) - g(\theta)) \xrightarrow{L} N(0, g'(\theta)^2\theta)$. So we want to find such a g that satisfies $g'(\theta)^2\theta = 1$. If we put $g(y) := 2y^{1/2}$ for $y \in [0, \infty)$, then $g'(\theta) = \theta^{-1/2}$ and $g'(\theta)^2\theta = 1$. Thus, $\sqrt{n}(2\bar{X}_n^{1/2} - 2\theta^{1/2}) \xrightarrow{L} N(0, 1)$.

1e. The approximate 95% confidence interval for $\theta^{1/2}$, namely $\bar{X}_n^{1/2} - 1.96/(2\sqrt{n}) \le \theta^{1/2} \le \bar{X}_n^{1/2} + 1.96/(2\sqrt{n})$, yields $[\bar{X}_n^{1/2} - 1.96/(2\sqrt{n})]^2 \le \theta \le [\bar{X}_n^{1/2} + 1.96/(2\sqrt{n})]^2$ as an approximate 95% confidence interval for θ.

2a. We have $E[X_1] = \int_0^1 x f(x;\theta)\, dx = \int_0^1 \theta x^\theta \, dx = \theta/(\theta+1)$ and $E[X_1^2] = \int_0^1 x^2 f(x;\theta)\, dx = \int_0^1 \theta x^{\theta+1}\, dx = \theta/(\theta+2)$. As such, $Var[X_1] = E[X_1^2] - E[X_1]^2 = \theta/(\theta+2) - \theta^2/(\theta+1)^2 = \theta/[(\theta+2)(\theta+1)^2]$.

2b. By the Central Limit Theorem, we have $\sqrt{n}(\bar{X}_n - E[X_1]) \xrightarrow{L} N(0, Var[X_1])$. Thus, putting $h(y) := y/(y+1)$ for $y \in (0, \infty)$, we have $\sqrt{n}(\bar{X}_n - h(\theta)) = \sqrt{n}(\bar{X}_n - \theta/(\theta+1)) \xrightarrow{L} N(0, \theta/[(\theta+2)(\theta+1)^2])$.

2c. Put $g(y) := y/(1-y)$ for $y \in (0,1)$. Note that $g'(y) = 1/(1-y)^2$ for $y \in (0,1)$. Then $g(\theta/(\theta+1)) = \theta$ and $g'(\theta/(\theta+1)) = (\theta+1)^2$. The delta method yields $\sqrt{n}(g(\bar{X}_n) - g(\theta/(\theta+1))) = \sqrt{n}(\bar{X}_n/(1-\bar{X}_n) - \theta) \xrightarrow{L} N(0, g'(\theta)^2\theta/[(\theta+2)(\theta+1)^2]) = N(0, \theta(\theta+1)^2/(\theta+2))$.

2d. Since $1/\sqrt{n} \xrightarrow{L} 0$, Slutsky's Theorem #3 yields $(\bar{X}_n/(1-\bar{X}_n) - \theta) = (1/\sqrt{n}) \times \sqrt{n}(\bar{X}_n/(1-\bar{X}_n) - \theta) \xrightarrow{L} 0 \times N(0, \theta(\theta+1)^2/(\theta+2)) = 0$. Since $\theta \xrightarrow{L} \theta$, another application of Slutsky's Theorem #3 yields $\bar{X}_n/(1-\bar{X}_n) = (\bar{X}_n/(1-\bar{X}_n) - \theta) + \theta \xrightarrow{L} 0 + \theta = \theta$. By the Continuous Mapping Theorem with $h(y) := (y(y+1)^2/(y+2))^{-1/2}$ for $y \in (0,\infty)$, we have $(\bar{X}_n/[(1-\bar{X}_n)^2(2-\bar{X}_n)])^{-1/2} = h(\bar{X}_n/(1-\bar{X}_n)) \xrightarrow{L} h(\theta) = (\theta(\theta+1)^2/(\theta+2))^{-1/2}$. Applying Slutsky's Theorem #3, we obtain $(\bar{X}_n/[(1-\bar{X}_n)^2(2-\bar{X}_n)])^{-1/2} \times \sqrt{n}(\bar{X}_n/(1-\bar{X}_n) - \theta) \xrightarrow{L} (\theta(\theta+1)^2/(\theta+2))^{-1/2} \times N(0, \theta(\theta+1)^2/(\theta+2)) = N(0,1)$. From this we find the approximate 95% confidence interval $\bar{X}_n/(1-\bar{X}_n) - 1.96\sqrt{\bar{X}_n/[n(1-\bar{X}_n)^2(2-\bar{X}_n)]} \le \theta \le \bar{X}_n/(1-\bar{X}_n) + 1.96\sqrt{\bar{X}_n/[n(1-\bar{X}_n)^2(2-\bar{X}_n)]}$.

2e. If $\bar{X}_n$ is close to 0, then θ is presumably close to 0 as well. As such, there is little uncertainty about θ, and so the confidence interval is relatively narrow. If $\bar{X}_n$ is close to 1, then θ is presumably "large". However, there is no finite upper limit to the possible values for θ, and so θ being "large" still allows for considerable uncertainty and hence a relatively wide confidence interval.