BST 676 — Spring 2011 — Dr. Charnigo

Written Assignment 2 Solutions

1a. The likelihood function is $L(\zeta;\mathbf{x}) := \prod_{i=1}^n \zeta(1-\zeta)^{x_i} = \zeta^n(1-\zeta)^{\sum_{i=1}^n x_i}$ for $\zeta \in (0,1]$. If $\sum_{i=1}^n x_i = 0$, then we have $L(\zeta;\mathbf{x}) = \zeta^n$, an increasing function of $\zeta \in (0,1]$ whose maximum on $(0,1]$ is attained at $\zeta = 1$. If $\sum_{i=1}^n x_i > 0$, then we differentiate the log likelihood $l(\zeta;\mathbf{x}) := n\log\zeta + \sum_{i=1}^n x_i \log(1-\zeta)$ with respect to ζ to obtain $n/\zeta - \sum_{i=1}^n x_i/(1-\zeta)$, whose unique root in $(0,1]$ occurs at $\zeta = 1/(\bar{x}+1)$. Since $\lim_{\zeta\to 0^+} l(\zeta;\mathbf{x}) = \lim_{\zeta\to 1^-} l(\zeta;\mathbf{x}) = -\infty$, we conclude that $1/(\bar{x}+1)$ is the global maximizer of the log likelihood function and therefore also of the likelihood function. Thus, the maximum likelihood estimate of θ is $\hat{\theta} := 1/(\bar{x}+1)$. (This formula also works when $\sum_{i=1}^n x_i = 0$ since then $1/(\bar{x}+1) = 1$.)

1b. The method of moments estimate $\hat{\theta}$ satisfies $\bar{x} = E_{\hat{\theta}}[X_1] = (1-\hat{\theta})/\hat{\theta}$. We solve this equation for $\hat{\theta}$ to obtain $\hat{\theta} = 1/(1+\bar{x})$. (So, in this example, the method of moments estimate coincides with the maximum likelihood estimate.)

1c. In all that follows $\theta \in (0,1]$. We have $p(\theta;\mathbf{x}) \propto p(\theta)L(\theta;\mathbf{x}) \propto \theta^{n+a-1}(1-\theta)^{\sum_{i=1}^n x_i + b - 1}$. The posterior mode is obtained by maximizing $p(\theta;\mathbf{x})$ with respect to $\theta \in (0,1]$. If $\sum_{i=1}^n x_i + b - 1 = 0$, then we have $p(\theta;\mathbf{x}) \propto \theta^{n+a-1}$, an increasing function of $\theta \in (0,1]$ whose maximum on $(0,1]$ is attained at $\theta = 1$. If $\sum_{i=1}^n x_i + b - 1 > 0$, then we differentiate $\log p(\theta;\mathbf{x}) = (n+a-1)\log\theta + (\sum_{i=1}^n x_i + b - 1)\log(1-\theta) + C$, where C is a quantity that does not depend on θ, with respect to θ to obtain $(n+a-1)/\theta - (\sum_{i=1}^n x_i + b - 1)/(1-\theta)$, whose unique root in $(0,1]$ occurs at $\theta = (n+a-1)/(n+a-1+\sum_{i=1}^n x_i + b - 1)$. Since $\lim_{\theta\to 0^+}\log p(\theta;\mathbf{x}) = \lim_{\theta\to 1^-}\log p(\theta;\mathbf{x}) = -\infty$, we conclude that $(n+a-1)/(n+a-1+\sum_{i=1}^n x_i + b - 1)$ is the global maximizer of $\log p(\theta;\mathbf{x})$ and therefore also of $p(\theta;\mathbf{x})$. Thus, the posterior mode is $(n+a-1)/(n+a-1+\sum_{i=1}^n x_i + b - 1)$. (This formula also works when $\sum_{i=1}^n x_i + b - 1 = 0$ since then $(n+a-1)/(n+a-1+\sum_{i=1}^n x_i + b - 1) = 1$.)

1d. In all that follows $\theta \in (0,1]$. Since $\int_0^1 p(\theta)\,d\theta = 1$, we know that $\int_0^1 \theta^{a-1}(1-\theta)^{b-1}\,d\theta = \Gamma[a]\Gamma[b]/\Gamma[a+b]$. Thus, we see that $\int_0^1 \theta^{n+a-1}(1-\theta)^{\sum_{i=1}^n x_i + b - 1}\,d\theta = \Gamma[n+a]\Gamma[\sum_{i=1}^n x_i + b]/\Gamma[n+a+\sum_{i=1}^n x_i + b]$, whence

$$p(\theta;\mathbf{x}) = \frac{\Gamma[n+a+\sum_{i=1}^n x_i + b]}{\Gamma[n+a]\Gamma[\sum_{i=1}^n x_i + b]}\theta^{n+a-1}(1-\theta)^{\sum_{i=1}^n x_i + b - 1}.$$

To find the posterior mean, we calculate

$$\begin{aligned}
\int_0^1 \theta p(\theta;\mathbf{x})\,d\theta &= \int_0^1 \frac{\Gamma[n+a+\sum_{i=1}^n x_i + b]}{\Gamma[n+a]\Gamma[\sum_{i=1}^n x_i + b]}\theta^{n+a+1-1}(1-\theta)^{\sum_{i=1}^n x_i + b - 1}\,d\theta \\
&= \frac{n+a}{n+a+\sum_{i=1}^n x_i + b}\int_0^1 \frac{\Gamma[n+a+\sum_{i=1}^n x_i + b + 1]}{\Gamma[n+a+1]\Gamma[\sum_{i=1}^n x_i + b]}\theta^{n+a+1-1}(1-\theta)^{\sum_{i=1}^n x_i + b - 1}\,d\theta \\
&= \frac{n+a}{n+a+\sum_{i=1}^n x_i + b}.
\end{aligned}$$

2a. We have $1 = \int_{\mathbb{R}} f(x;\theta)\,dx = \int_{-\theta}^{\theta} C(\theta)\exp[x]\,dx = C(\theta)(\exp[\theta] - \exp[-\theta])$, whence $C(\theta) = (\exp[\theta] - \exp[-\theta])^{-1}$.

2b. The likelihood function is $L(\zeta;\mathbf{x}) := \prod_{i=1}^n f(x_i;\zeta) = (\exp[\theta] - \exp[-\theta])^{-n}\exp[\sum_{i=1}^n x_i]1_{\{|x_1|\le\zeta,\ldots,|x_n|\le\zeta\}} = (\exp[\theta] - \exp[-\theta])^{-n}\exp[\sum_{i=1}^n x_i]1_{\{\max_{1\le i\le n}|x_i|\le\zeta\}}$ for $\zeta \in (0,\infty)$. Since $\exp[\zeta] - \exp[-\zeta]$ is an increasing function of $\zeta \in (0,\infty)$, we see that $(\exp[\theta] - \exp[-\theta])^{-n}$ is a decreasing function of $\zeta \in (0,\infty)$. As such, $L(\zeta;\mathbf{x})$ will be maximized when ζ is as small as possible without making the indicator $1_{\{\max_{1\le i\le n}|x_i|\le\zeta\}}$ vanish. In other words, the maximum likelihood estimate is $\hat{\theta} := \max_{1\le i\le n}|x_i|$.

2c. If $\bar{x} > 0$, then the method of moments estimate $\hat{\theta}$ satisfies

$$\begin{aligned}
\bar{x} &= E_{\hat{\theta}}[X_1] = \int_{-\hat{\theta}}^{\hat{\theta}} (\exp[\hat{\theta}] - \exp[-\hat{\theta}])^{-1} x \exp[x] \; dx \\
&= (\exp[\hat{\theta}] - \exp[-\hat{\theta}])^{-1} \{x \exp[x]|_{-\hat{\theta}}^{\hat{\theta}} - \int_{-\hat{\theta}}^{\hat{\theta}} \exp[x] \; dx\} \\
&= (\exp[\hat{\theta}] - \exp[-\hat{\theta}])^{-1} \{\hat{\theta} \exp[\hat{\theta}] + \hat{\theta} \exp[-\hat{\theta}] - \exp[\hat{\theta}] + \exp[-\hat{\theta}]\} \\
&= \hat{\theta}(\exp[\hat{\theta}] + \exp[-\hat{\theta}])/(\exp[\hat{\theta}] - \exp[-\hat{\theta}]) - 1 \\
&= \hat{\theta} \coth[\hat{\theta}] - 1.
\end{aligned}$$

Otherwise, the method of moments estimate does not exist since $\zeta \coth[\zeta] - 1 > 0$ for all $\zeta \in (0, \infty)$.

2d. The following R code produced the figure shown below.

```
postscript("C:\\Documents and Settings\\Rich\\Desktop\\Rich\\BST676S11\\WA2676S11Graph.ps")
hattheta <- (0:600)/100
barx <- hattheta*cosh(hattheta)/sinh(hattheta)-1
plot(barx,hattheta,xlim=c(0,5),xlab="Sample mean",ylab="Method of moments estimate")
title("Exercise 2d")
dev.off()
```

Figure 1: