BST 676 — Spring 2011 — Dr. Charnigo

Written Assignment 3

Written Assignment 3 is due on Thursday 03 March at the end of class. You are encouraged to work in groups of two or three, but you may work individually if you prefer.

Suppose that $X_1, X_2, \ldots, X_n \overset{iid}{\sim} f(x;\theta) := \theta^x (1-\theta)^{1-x}$ for $x \in \{0,1\}$, where $\theta \in \Theta := (0,1)$.

[10] 1. Evaluate the Cramer-Rao lower bound for unbiased estimation of θ.

[10] 2. Find the best unbiased estimator of θ. For subsequent items, let the best unbiased estimator be denoted $\hat{\theta}$.

[10] 3. Find $a \in (0,\infty)$, possibly depending on n and on θ, for which the mean square error of $a\hat{\theta}$ is minimal.

Remark. Since θ is unknown in practice, a data analyst cannot identify this optimal a. One way to address this difficulty is for the data analyst to choose a to minimize not $E_\theta[(a\hat{\theta}-\theta)^2]$ but rather $\int_0^1 E_\theta[(a\hat{\theta}-\theta)^2]\, p(\theta)\, d\theta$ for some function $p(\theta)$ that may be regarded as a prior probability density for θ.

[10] 4. One may contend that $\hat{\theta}$ being too close to 0 or too close to 1 is especially bad. As such, mean square error may not be an ideal criterion for estimator evaluation. Propose another criterion $E_\theta[L(\theta,\hat{\theta})]$, where $L(a,b)$ is a function with domain $(0,1)\times(0,1)$ satisfying the following properties: (i) $L(a,b) \geq 0$; (ii) for fixed a, $L(a,b)$ is decreasing in b when $b < a$ and increasing in b when $b > a$; and, (iii) for fixed a, $\lim_{b\to 0^+} L(a,b) = \lim_{b \to 1^-} L(a,b) = +\infty$.

[10] 5. Describe the large-sample behavior of $\sqrt{n}(\hat{\theta}^2 - \theta^2)$.

[10] 6. Evaluate $E_\theta[\hat{\theta}^2]$. Note that, although $\hat{\theta}^2$ is asymptotically unbiased for θ^2 per exercise 5, there is some finite-sample bias.

[10] 7. Evaluate the Cramer-Rao lower bound for unbiased estimation of $\tau := \theta^2$.

[10] 8. Find $a, b \in (-\infty,\infty)$, possibly depending on n but not on θ, such that $\hat{\tau} := a\hat{\theta}^2 + b\hat{\theta}$ is unbiased for τ.

[10] 9. Without actually calculating the variance of $\hat{\tau}$, argue that the variance of $\hat{\tau}$ must exceed the Cramer-Rao lower bound for unbiased estimation of τ.

[10] 10. Prove that $\hat{\tau}$ is consistent for τ.