BST 676 — Spring 2011 — Dr. Charnigo

Written Assignment 3 Solutions

1. For $x \in \{0,1\}$ we have $\log f(x;\theta) = x\log\theta + (1-x)\log(1-\theta)$ and $\frac{\partial}{\partial\theta}\log f(x;\theta) = x/\theta + (x-1)/(1-\theta) = x/[\theta(1-\theta)] - 1/(1-\theta)$. Thus, $J_n(\theta) = nVar_\theta\left[\frac{\partial}{\partial\theta}\log f(X;\theta)\right] = nVar_\theta\left[\frac{X}{\theta(1-\theta)}\right] = n\theta^{-2}(1-\theta)^{-2}Var_\theta[X] = n\theta^{-1}(1-\theta)^{-1}$. The Cramer-Rao lower bound for unbiased estimation of θ is $1/J_n(\theta) = \theta(1-\theta)/n$.

2. We have $E_\theta[\bar{X}] = E_\theta[X_1] = \theta$ and $Var_\theta[\bar{X}] = Var_\theta[X_1]/n = \theta(1-\theta)/n$, which is the Cramer-Rao lower bound for unbiased estimation of θ. Thus, $\bar{X}$ is the best unbiased estimator of θ.

3. The mean square error of $a\hat{\theta}$ is $(E_\theta[a\hat{\theta}] - \theta)^2 + Var_\theta[a\hat{\theta}] = (a-1)^2\theta^2 + a^2\theta(1-\theta)/n$. The derivative of the mean square error with respect to a is $2(a-1)\theta^2 + 2a\theta(1-\theta)/n$. Setting the derivative to 0 yields $a = \theta^2/[\theta^2 + \theta(1-\theta)/n] = 1/[1 + (1-\theta)/(n\theta)]$. Since the derivative is a linear function of a, we see that the derivative must be negative for $a < 1/[1 + (1-\theta)/(n\theta)]$ and positive for $a > 1/[1 + (1-\theta)/(n\theta)]$. Hence, setting the derivative to 0 yields the global minimizer. In summary, the optimal a is $1/[1 + (1-\theta)/(n\theta)]$.

Remark. For a fixed n, if θ is close to 1, then the optimal a is close to 1. For a fixed n, if θ is close to 0, then the optimal a is considerably smaller than 1. For a fixed θ, the optimal a tends to 1 as n becomes large.

4. Many answers are possible. One is $L(a,b) := (\text{logit } b - \text{logit } a)^2$. Condition (i) is obviously satisfied. To check condition (ii), note that $\frac{\partial}{\partial b}L(a,b) = 2(\text{logit } b - \text{logit } a)\frac{\partial}{\partial b}\text{logit } b = 2(\text{logit } b - \text{logit } a)[1/b + 1/(1-b)]$. This is negative if $b < a$ (since then $\text{logit } b < \text{logit } a$) and positive if $b > a$ (since then $\text{logit } b > \text{logit } a$), implying that $L(a,b)$ is decreasing in b when $b < a$ but increasing in b when $b > a$. Condition (iii) follows from $\lim_{b\to 0^+}\text{logit } b = -\infty$ and $\lim_{b\to 1^-}\text{logit } b = +\infty$.

5. The Central Limit Theorem provides $\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{L} N(0, \theta(1-\theta))$. Applying the Delta Method with $g(y) := y^2$ and $g'(y) = 2y$, we find that $\sqrt{n}(\hat{\theta}^2 - \theta^2) \xrightarrow{L} 2\theta N(0,\theta(1-\theta)) = N(0, 4\theta^3(1-\theta))$.

6. We have $E_\theta[\hat{\theta}^2] = Var_\theta[\hat{\theta}] + \left(E_\theta[\hat{\theta}]\right)^2 = \theta(1-\theta)/n + \theta^2$.

7. The Cramer-Rao lower bound for unbiased estimation of θ^2 is $\left(\frac{d}{d\theta}\theta^2\right)^2/J_n(\theta) = 4\theta^2/[n\theta^{-1}(1-\theta)^{-1}] = 4\theta^3(1-\theta)/n$.

8. We have $E_\theta[a\hat{\theta}^2 + b\hat{\theta}] = a\theta(1-\theta)/n + a\theta^2 + b\theta = a(1-1/n)\theta^2 + (b + a/n)\theta$. Thus, we need $a(1-1/n) = 1$ and $b + a/n = 0$ for unbiased estimation of θ^2, yielding $a = n/(n-1)$ and $b = -1/(n-1)$.

9. In general, if $\nu_1(\theta)$ and $\nu_2(\theta)$ are not linearly related, then we cannot attain both the Cramer-Rao lower bound for unbiased estimation of $\nu_1(\theta)$ and the Cramer-Rao lower bound for unbiased estimation of $\nu_2(\theta)$. For this exercise, put $\nu_1(\theta) := \theta$ and $\nu_2(\theta) := \theta^2$. Since the Cramer-Rao lower bound for unbiased estimation of θ was attained (exercise 2), the Cramer-Rao lower bound for unbiased estimation of θ^2 cannot be attained.

10. Note that $a \to 1$ and $b \to 0$ as $n \to \infty$. Also, note that $\hat{\theta} \xrightarrow{P} \theta$ by the Weak Law and $\hat{\theta}^2 \xrightarrow{P} \theta^2$ by the Continuous Mapping Theorem. Thus, Slutsky's Theorem #2 yields $a \times \hat{\theta}^2 \xrightarrow{P} 1 \times \theta^2 = \theta^2$ and $b \times \hat{\theta} \xrightarrow{P} 0 \times \theta = 0$. Another application of Slutsky's Theorem #2 yields $a\hat{\theta}^2 + b\hat{\theta} \xrightarrow{P} \theta^2 + 0 = \theta^2$.