BST 676 — Spring 2011 — Dr. Charnigo

Written Assignment 4 Solutions

1. The tables relevant to performing Fisher's exact test are of the form

	High Urgency	Low Urgency	Row Total
Responsive to Stimulus	x	$13-x$	13
Unresponsive to Stimulus	$10-x$	$x-3$	7
Column Total	10	10	20

for $x \in \{3,4,5,6,7,8,9,10\}$.

The probabilities attached to these tables are 0.001547988, 0.027089783, 0.146284830, 0.325077399, 0.325077399, 0.146284830, 0.027089783, and 0.001547988 respectively. So, the p-value for Fisher's exact test of $H_0: p_1 = p_2$ against $H_1: p_1 \neq p_2$ is $0.001547988+0.027089783+0.027089783+0.001547988 = 0.05727554$ and we narrowly avoid rejecting H_0. This agrees with the result I obtained from SAS, for which I used the code below.

```
data Trial;
input HighUrgency Responsive Count;
datalines;
0 0 6
1 0 1
0 1 4
1 1 9
run;
proc freq data=Trial;
weight Count;
tables HighUrgency*Responsive / chisq;
run;
```

2a. Put $\widehat{\theta} := \bar{X}$. Then $\widehat{\theta}$ is unbiased with variance θ^2/n and asymptotically normal (by the Central Limit Theorem), so that

$$\frac{\widehat{\theta}-\theta}{\sqrt{\theta^2/n}} \xrightarrow{L} N(0,1).$$

Since $\widehat{\theta}$ is consistent for θ (by the Weak Law of Large Numbers), the Continuous Mapping Theorem yields

$$\frac{\sqrt{\theta^2/n}}{\sqrt{\widehat{\theta}^2/n}} = \frac{\theta}{\widehat{\theta}} \xrightarrow{P} 1,$$

whence Slutsky's Theorem #3 provides

$$\frac{\widehat{\theta}-\theta}{\sqrt{\widehat{\theta}^2/n}} \xrightarrow{L} N(0,1).$$

If H_0 is true, then

$$\frac{\widehat{\theta}-\theta_0}{\sqrt{\widehat{\theta}^2/n}} \xrightarrow{L} N(0,1), \tag{1}$$

so that a Wald test may be defined by rejecting H_0 if the left member of (1) exceeds $z_{1-\alpha/2}$ in absolute value.

2b. The score statistic is

$$S(\theta; \mathbf{X}) := \sum_{i=1}^{n} \frac{\partial}{\partial\theta} \log f(X_i; \theta) = -n/\theta + \sum_{i=1}^{n} X_i/\theta^2,$$

whose variance is $J_n(\theta) = n/\theta^2$. If H_0 is true, then

$$\frac{\sum_{i=1}^{n} X_i/\theta_0^2 - n/\theta_0}{\sqrt{n/\theta_0^2}} = \frac{\widehat{\theta} - \theta_0}{\sqrt{\theta_0^2/n}} \xrightarrow{L} N(0,1), \tag{2}$$

so that a score test may be defined by rejecting H_0 if the left member of (2) exceeds $z_{1-\alpha/2}$ in absolute value.

3a. From Example #5 of Unit II we know that $L(\zeta; \mathbf{X}) = \zeta^{-n} 1_{\{\zeta \geq \max_{1\leq i\leq n} X_i\}}$ for $\zeta \in (0, \infty)$ and that the maximum likelihood estimator is $\hat{\theta} := \max_{1\leq i\leq n} X_i$. Noting that $1_{\{\theta_0 \geq \max_{1\leq i\leq n} X_i\}} = 1_{\{\theta_0 \geq \hat{\theta}\}} = 1_{\{\hat{\theta}/\theta_0 \leq 1\}}$ and that $1_{\{\hat{\theta} \geq \max_{1\leq i\leq n} X_i\}} = 1_{\{\hat{\theta} \geq \hat{\theta}\}} = 1$, we see that the likelihood ratio test statistic $\lambda := L(\theta_0; \mathbf{X})/L(\hat{\theta}; \mathbf{X})$ is $(\hat{\theta}/\theta_0)^n 1_{\{\hat{\theta}/\theta_0 \leq 1\}}$.

3b. For $a \in [0, \theta_0]$ we have $P_{\theta_0}[\hat{\theta} \leq a] = P_{\theta_0}[\cap_{i=1}^{n} (X_i \leq a)] = \prod_{i=1}^{n} P_{\theta_0}[X_i \leq a] = \prod_{i=1}^{n} \int_0^a 1/\theta_0 \; dx_i = (a/\theta_0)^n$.

3c. For $a \in [0, 1]$ we have $P_{\theta_0}[(\hat{\theta}/\theta_0)^n \leq a] = P_{\theta_0}[\hat{\theta} \leq \theta_0 a^{1/n}] = (\theta_0 a^{1/n}/\theta_0)^n = a$, the second last equality following from exercise 3b.

3d. Put $c := 0.05$. Then rejection of H_0 with $\lambda < c$ yields a test at significance level 0.05 because $P_{\theta_0}[\lambda < 0.05] = P_{\theta_0}[(\hat{\theta}/\theta_0)^n 1_{\{\hat{\theta}/\theta_0 \leq 1\}} < 0.05] = P_{\theta_0}[(\hat{\theta}/\theta_0)^n < 0.05] = P_{\theta_0}[(\hat{\theta}/\theta_0)^n \leq 0.05] = 0.05$, the last equality following from exercise 3c. The second equality holds because the indicator equals 1 with probability one if $\theta = \theta_0$. The second last equality holds because, as is evident from exercise 3b, $\hat{\theta}$ is a continuous random variable.