BST 676 — Spring 2011 — Dr. Charnigo

Written Assignment 5 Solutions

1a. Suppose $\theta = \zeta < \theta_0$. The indicator $1_{\{\hat{\theta}/\theta_0 \leq 1\}}$ equals 1 since $\hat{\theta} \leq \theta = \zeta < \theta_0$. As such, we have

$$\begin{aligned}
G(\zeta) &= P_\zeta[\lambda < 0.05] \\
&= P_\zeta[(\hat{\theta}/\theta_0)^n 1_{\{\hat{\theta}/\theta_0 \leq 1\}} < 0.05] \\
&= P_\zeta[(\hat{\theta}/\theta_0)^n < 0.05] \\
&= P_\zeta[\hat{\theta} < 0.05^{1/n}\theta_0] \\
&= (P_\zeta[X_1 < 0.05^{1/n}\theta_0])^n \\
&= (\min\{0.05^{1/n}\theta_0/\zeta, 1\})^n \\
&= \min\{0.05\theta_0^n/\zeta^n, 1\}.
\end{aligned}$$

1b. Suppose $\theta = \zeta > \theta_0$. If $(\hat{\theta}/\theta_0)^n 1_{\{\hat{\theta}/\theta_0 \leq 1\}} < 0.05$, then either the indicator $1_{\{\hat{\theta}/\theta_0 \leq 1\}}$ equals 0 or $(\hat{\theta}/\theta_0)^n < 0.05$ (in which case the indicator equals 1, so that these two contingencies are mutually exclusive). As such, we have

$$\begin{aligned}
G(\zeta) &= P_\zeta[\lambda < 0.05] \\
&= P_\zeta[(\hat{\theta}/\theta_0)^n 1_{\{\hat{\theta}/\theta_0 \leq 1\}} < 0.05] \\
&= P_\zeta[(\hat{\theta}/\theta_0)^n < 0.05] + P_\zeta[1_{\{\hat{\theta}/\theta_0 \leq 1\}} = 0] \\
&= P_\zeta[\hat{\theta} < 0.05^{1/n}\theta_0] + 1 - P_\zeta[\hat{\theta} \leq \theta_0] \\
&= (P_\zeta[X_1 < 0.05^{1/n}\theta_0])^n + 1 - (P_\zeta[X_1 \leq \theta_0])^n \\
&= (0.05^{1/n}\theta_0/\zeta)^n + 1 - (\theta_0/\zeta)^n \\
&= 1 - 0.95\theta_0^n/\zeta^n.
\end{aligned}$$

1c. If $\zeta < \theta_0$, then $\theta_0^n/\zeta^n > 1$ so that $0.05\theta_0^n/\zeta^n > 0.05$ and hence $G(\zeta) = \min\{0.05\theta_0^n/\zeta^n, 1\} > 0.05 = G(\theta_0)$. If $\zeta > \theta_0$, then $\theta_0^n/\zeta^n < 1$ so that $0.95\theta_0^n/\zeta^n < 0.95$ and hence $G(\zeta) = 1 - 0.95\theta_0^n/\zeta^n > 0.05 = G(\theta_0)$. Hence, the likelihood ratio test is unbiased.

1d. If $\zeta < \theta_0$, then $\lim_{n\to\infty} \theta_0^n/\zeta^n = \infty$. As such, $\lim_{n\to\infty} G(\zeta) = \lim_{n\to\infty} \min\{0.05\theta_0^n/\zeta^n, 1\} = 1$. If $\zeta > \theta_0$, then $\lim_{n\to\infty} \theta_0^n/\zeta^n = 0$. As such, $\lim_{n\to\infty} G(\zeta) = \lim_{n\to\infty}(1 - 0.95\theta_0^n/\zeta^n) = 1$. Hence, the likelihood ratio test is consistent.

2a. We have

$$\begin{aligned}
S(\theta; \mathbf{X}) &= \sum_{i=1}^n \frac{\partial}{\partial\theta} \log f(X_i; \theta) \\
&= \sum_{i=1}^n \frac{\partial}{\partial\theta}\{-(1/2)\log(2\pi) - (1/2)\log\theta - X_i^2/(2\theta)\} \\
&= \sum_{i=1}^n \{-(1/2)/\theta + X_i^2/(2\theta^2)\} \\
&= -n/(2\theta) + \sum_{i=1}^n X_i^2/(2\theta^2)
\end{aligned}$$

and, noting that $X_i^2/\theta \sim \chi_1^2$,

$$J_n(\theta) = Var_\theta\,[S(\theta; \mathbf{X})]$$

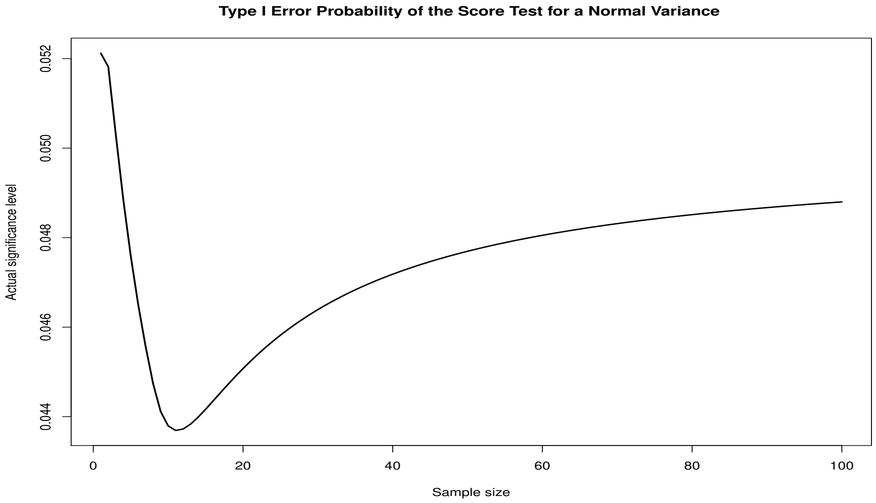
$$
\begin{aligned}
&= Var_\theta\left[-n/(2\theta) + \sum_{i=1}^n X_i^2/(2\theta^2)\right] \\
&= Var_\theta\left[\sum_{i=1}^n X_i^2/(2\theta^2)\right] \\
&= \sum_{i=1}^n Var_\theta[X_i^2/(2\theta^2)] \\
&= \sum_{i=1}^n Var_\theta[X_i^2/(\theta \times 2\theta)] \\
&= \sum_{i=1}^n Var_\theta[X_i^2/\theta]/(2\theta)^2 \\
&= \sum_{i=1}^n 2/(2\theta)^2 \\
&= n/(2\theta^2).
\end{aligned}
$$

Thus, the score test statistic is

$$
\begin{aligned}
Z &:= S(\theta_0; \mathbf{X})/\sqrt{J_n(\theta_0)} \\
&= \left\{-n/(2\theta_0) + \sum_{i=1}^n X_i^2/(2\theta_0^2)\right\}/\sqrt{n/(2\theta_0^2)} \\
&= \sqrt{n/2}\left\{\sum_{i=1}^n X_i^2/(n\theta_0) - 1\right\}.
\end{aligned}
$$

Rejection of $H_0 : \theta = \theta_0$ in favor of $H_1 : \theta \neq \theta_0$ at approximate significance level α occurs when $|Z| > 1.96$.

Figure 1:



2b. If H_0 is true, $\sum_{i=1}^n X_i^2/\theta_0 \sim \chi_n^2$. Since $|Z| > 1.96$ if and only if either $\sum_{i=1}^n X_i^2/\theta_0 > n + 1.96\sqrt{2n}$ or $\sum_{i=1}^n X_i^2/\theta_0 < n - 1.96\sqrt{2n}$, the actual significance level of the score test is $F_n(n - 1.96\sqrt{2n}) + 1 -$

$F_n(n+1.96\sqrt{2n})$, where F_n denotes the χ^2_n cumulative distribution function. The plot of the actual significance level as a function of $n \in \{1, 2, \ldots, 100\}$ reveals that the actual significance level is bounded between 0.0437 and 0.0521. If one wishes that the actual significance level should be within 0.0100 of the nominal significance level, then the score test may be comfortably used at any sample size. If one wishes that the actual significance level should be within 0.0050 of the nominal significance level, then the score test may be comfortably used at a sample size of 20 or greater.

2c. We have $f(x;\theta) = a(x)b(\theta)\exp[c(x)d(\theta)]$ with $a(x) = 1$, $b(\theta) = (2\pi\theta)^{-1/2}$, $c(x) = x^2$, and $d(\theta) = -1/(2\theta)$. Put $T := \sum_{i=1}^n c(X_i) = \sum_{i=1}^n X_i^2$. Since $T/\theta \sim \chi^2_n = Gamma(n/2, 2)$, we have $T \sim Gamma(n/2, 2\theta)$. Thus, $h(t;\theta) = \frac{1}{\Gamma[n/2](2\theta)^{n/2}} t^{n/2-1}\exp[-t/(2\theta)]$ for $t > 0$. As such, $h(t;\theta_1)/h(t;\theta_2) = \exp[-0.5t\{1/\theta_1 - 1/\theta_2\}]\theta_2^{n/2}/\theta_1^{n/2}$ for $t > 0$, which is increasing in t when $\theta_1 > \theta_2$ since then $-0.5(1/\theta_1 - 1/\theta_2) > 0$. As such, a uniformly most powerful level 0.05 test of $H_0 : \theta \geq \theta_0$ against $H_1 : \theta < \theta_0$ is obtained by rejecting the null hypothesis when T is less than the 0.05 quantile of its distribution under θ_0, namely $Gamma(n/2, 2\theta_0)$. This is equivalent to rejecting the null hypothesis when T/θ_0 is less than the 0.05 quantile of χ^2_n. In contrast, the score test rejected its null hypothesis when T/θ_0 was less than the 0.025 quantile of $N(n, 2n)$, an approximation to χ^2_n with the same mean and variance, or greater than the 0.975 quantile of $N(n, 2n)$. Thus, we may regard the uniformly most powerful test as an exact one-sided version of the score test.