STA 580 — Fall 2008 — Dr. Charnigo

Written Assignment 3 Solutions

1a. Using notation as in Lecture 7, the test statistic is

$$\chi^2 = \frac{(n_1-1)s_x^2}{100} = \frac{(73)841.73}{100} = 614.46.$$

The critical value is

$$\chi^2_{n_1-1,1-\alpha} = \chi^2_{73,0.95} = 93.95.$$

Since $614.46 > 93.95$ we (easily!) reject $H_0 : \sigma_1^2 = 100$ in favor of $H_1 : \sigma_1^2 > 100$.

1b. We have

$$s^2 = \frac{(n_1-1)s_x^2 + (n_2-1)s_y^2}{n_1+n_2-2} = \frac{(73)841.73 + (125)565.57}{198} = 667.39.$$

The test statistic is

$$t = \frac{\bar{x}-\bar{y}}{\sqrt{s^2(1/n_1+1/n_2)}} = \frac{149.84 - 114.41}{\sqrt{667.39(1/74+1/126)}} = \frac{35.43}{3.784} = 9.363.$$

The (upper) critical value is

$$t_{n_1+n_2-2,1-\alpha/2} = t_{198,0.975} = 1.972.$$

Since $9.363 > 1.972$ we (easily!) reject $H_0 : \mu_1 = \mu_2$ in favor of $H_1 : \mu_1 \neq \mu_2$.

1c. We have

$$df = \frac{(s_x^2/n_1 + s_y^2/n_2)^2}{(s_x^2/n_1)^2/(n_1-1) + (s_y^2/n_2)^2/(n_2-1)} = \frac{(841.73/74 + 565.57/126)^2}{(841.73/74)^2/(73) + (565.57/126)^2/(125)} = \frac{251.65}{1.934} \approx 130.$$

The test statistic is

$$t = \frac{\bar{x}-\bar{y}}{\sqrt{s_x^2/n_1 + s_y^2/n_2}} = \frac{149.84 - 114.41}{\sqrt{841.73/74 + 565.57/126}} = \frac{35.43}{3.983} = 8.895.$$

The (upper) critical value is

$$t_{df,1-\alpha/2} \approx t_{130,0.975} = 1.978.$$

Since $8.895 > 1.978$ we (easily!) reject $H_0 : \mu_1 = \mu_2$ in favor of $H_1 : \mu_1 \neq \mu_2$.

1d. The test statistic is

$$f = \frac{s_x^2}{s_y^2} = \frac{841.73}{565.57} = 1.488.$$

The upper critical value is

$$f_{n_1-1,n_2-1,1-\alpha/2} = f_{73,125,0.975} = 1.491,$$

while the lower critical value is

$$f_{n_1-1,n_2-1,\alpha/2} = f_{73,125,0.025} = 0.656.$$

[This lower critical value can also be written as $1/1.525 = 1/f_{125,73,0.975} = 1/f_{n_2-1,n_1-1,1-\alpha/2}$.] Since $0.656 < 1.488 < 1.491$ we accept $H_0 : \sigma_1^2 = \sigma_2^2$. Given that we have accepted $H_0 : \sigma_1^2 = \sigma_2^2$, most people would say that the test in part b is more appropriate for comparing means than the test in part c.

1e. For part a we could not obtain an answer without the normality assumption. For parts b and c we could perform the large sample ("z") test on page 5 of Lecture 7, although there is no provision in this test to impose an assumption that $\sigma_1^2 = \sigma_2^2$. For part d we could not obtain an answer without the normality assumption. In summary we have either (iii) (ii) (ii) (iii) or (iii) (iii) (ii) (iii), depending on whether one balks at the large sample test's inability to impose an assumption that $\sigma_1^2 = \sigma_2^2$.

2a. Using notation as in Lecture 6, the test statistic is

$$z = \frac{\hat{p} - 0.3}{\sqrt{0.3(1-0.3)/n}} = \frac{74/200 - 0.3}{\sqrt{0.3(0.7)/200}} = \frac{0.07}{0.0324} = 2.16.$$

The (upper) critical value is

$$z_{1-\alpha/2} = z_{0.975} = 1.96.$$

Since $2.16 > 1.96$ we reject $H_0 : p = 0.3$ in favor of $H_1 : p \neq 0.3$.

2b. The power with a sample size of 500 is

$$\Phi\left[\sqrt{\frac{p_0(1-p_0)}{p_1(1-p_1)}}\left(-z_{1-\alpha/2} + \frac{\sqrt{n}|p_0 - p_1|}{\sqrt{p_0(1-p_0)}}\right)\right] = \Phi\left[\sqrt{\frac{0.3(0.7)}{74/200(126/200)}}\left(-1.96 + \frac{\sqrt{500}|0.3 - 74/200|}{\sqrt{0.3(0.7)}}\right)\right] =$$

$$\Phi\left[0.9492(1.456)\right] = \Phi[1.382] = 0.917.$$

2c. For 80% power the sample size is

$$\frac{p_0(1-p_0)\left(z_{1-\alpha/2} + z_{1-\beta}\sqrt{\frac{p_1(1-p_1)}{p_0(1-p_0)}}\right)^2}{(p_1 - p_0)^2} = \frac{0.3(0.7)\left(1.96 + 0.842\sqrt{\frac{74/200(126/200)}{0.3(0.7)}}\right)^2}{(74/200 - 0.3)^2} \approx 348.$$

For 90% power the sample size is

$$\frac{0.3(0.7)\left(1.96 + 1.282\sqrt{\frac{74/200(126/200)}{0.3(0.7)}}\right)^2}{(74/200 - 0.3)^2} \approx 470.$$

2d. The power with a sample size of $348 + 75 = 423$ is

$$\Phi\left[\sqrt{\frac{0.3(0.7)}{74/200(126/200)}}\left(-1.96 + \frac{\sqrt{423}|0.3 - 74/200|}{\sqrt{0.3(0.7)}}\right)\right] = \Phi\left[0.9492(1.182)\right] = \Phi[1.122] = 0.869.$$

Since the gain in power exceeds 5% we will increase the sample size from 348 to 423.

The power with a sample size of $470 + 75 = 545$ is

$$\Phi\left[\sqrt{\frac{0.3(0.7)}{74/200(126/200)}}\left(-1.96 + \frac{\sqrt{545}|0.3 - 74/200|}{\sqrt{0.3(0.7)}}\right)\right] = \Phi\left[0.9492(1.606)\right] = \Phi[1.524] = 0.936.$$

Since the gain in power is less than 5% we will not increase the sample size from 470 to 545.

2e. Moving from 90% power to 95% power, for example, requires a much greater increment in the sample size than moving from 80% power to 85% power. Thus, most researchers feel that aiming for more than 90% power is going well past a point of diminishing returns on their investment of study resources.