STA 580 — Spring 2009 — Dr. Charnigo

Written Assignment 2 Solutions

1a. From Written Assignment 1 we know that $\bar{x} = 3.389$, $s = 0.755$, and $n = 23$. The $100(1-\alpha)\%$ "small sample" confidence interval for μ is

$$\bar{x} \pm t_{n-1,1-\alpha/2} s/\sqrt{n}.$$

Putting $\alpha = 0.05$ and noting that $t_{n-1,1-\alpha/2} = t_{22,0.975} = 2.074$, we obtain

$$3.389 \pm 0.327, \text{ which is } 3.062 \text{ to } 3.716.$$

We could check whether the normality assumption seemed reasonable by constructing a histogram (or, alternatively, a stem-and-leaf display) and seeing whether the bars formed a bell-shaped pattern. [We could also check a box plot for symmetry, because a normal distribution is symmetric about its mean. However, a disadvantage of checking a box plot for symmetry is that not every symmetric distribution is normal, and distinguishing a normal distribution from a symmetric nonnormal distribution is difficult if one uses only a box plot.]

1b. Since 3.50 is contained in the 95% confidence interval, 3.50 is a plausible value for μ. So we do not anticipate rejecting $H_0 : \mu = 3.50$ in favor of $H_1 : \mu \neq 3.50$. Formally, we conduct a level α "small sample" test of $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$ by constructing the test statistic

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

and comparing its absolute value to $t_{n-1,1-\alpha/2}$. In this case, with $\mu_0 = 3.50$ and $\alpha = 0.05$, we have

$$t = \frac{3.389 - 3.50}{0.755/\sqrt{23}} = \frac{-0.111}{0.157} = -0.707,$$

whose absolute value is less than $t_{22,0.975} = 2.074$. Therefore we do not reject $H_0 : \mu = 3.50$ in favor of $H_1 : \mu \neq 3.50$.

1c. The general formula for (approximate) power in testing $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$ at level α is

$$\Phi\left(-z_{1-\alpha/2} + \frac{|\mu_0 - \mu_1|\sqrt{n}}{\sigma}\right).$$

We have $\alpha = 0.05$, so that $-z_{1-\alpha/2} = -z_{0.975} = -1.960$. We also have $\mu_0 = 3.50$ and $n = 50$. Taking $\mu_1 = 3.389$ and $\sigma = 0.755$, as we have no compelling reason to do otherwise, we find that the power is

$$\Phi\left(-1.960 + \frac{|3.389 - 3.50|\sqrt{50}}{0.755}\right) = \Phi(-0.920) = 17.9\%.$$

Remark. The 17.9% is quite small but intuitively plausible. If μ really were 3.389, then the null hypothesis would almost be true. Rejecting a null hypothesis that is almost true requires an immense amount of data, and $n = 50$ is not an immense sample size.

1d. The general formula for (approximate) sample size in testing $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$ at level α, when one desires power $1 - \beta$, is

$$n = \frac{\sigma^2(z_{1-\beta} + z_{1-\alpha/2})^2}{(\mu_0 - \mu_1)^2}.$$

With $1 - \beta = 0.80$ we have $z_{1-\beta} = z_{0.80} = 0.842$. So the required sample size is

$$\frac{0.755^2(0.842 + 1.960)^2}{(3.50 - 3.389)^2} \approx 364.$$

1e. The $100(1 - \alpha)\%$ confidence interval for σ^2 is

$$\frac{(n-1)s^2}{\chi^2_{n-1,1-\alpha/2}} \text{ to } \frac{(n-1)s^2}{\chi^2_{n-1,\alpha/2}}.$$

Putting $\alpha = 0.05$ and noting that $\chi^2_{n-1,1-\alpha/2} = \chi^2_{22,0.975} = 36.78$, $\chi^2_{n-1,\alpha/2} = \chi^2_{22,0.025} = 10.98$, we obtain

$$\frac{(22)0.755^2}{36.78} \text{ to } \frac{(22)0.755^2}{10.98}, \text{ which is } 0.341 \text{ to } 1.142.$$

We could not abandon the normality assumption if the sample size were 500 because a normal population is required for the formula

$$\frac{(n-1)s^2}{\chi^2_{n-1,1-\alpha/2}} \text{ to } \frac{(n-1)s^2}{\chi^2_{n-1,\alpha/2}}$$

to be valid, regardless of n.

2a. We can easily calculate that $\bar{x} = 71.26$, $s = 11.48$, and $n = 200$. The $100(1 - \alpha)\%$ "large sample" confidence interval for μ is

$$\bar{x} \pm z_{1-\alpha/2}s/\sqrt{n}.$$

Putting $\alpha = 0.05$ and noting that $z_{1-\alpha/2} = z_{0.975} = 1.960$, we obtain

$$71.26 \pm 1.59, \text{ which is } 69.67 \text{ to } 72.85.$$

2b. We conduct a level α "large sample" test of $H_0 : \mu = \mu_0$ against $H_1 : \mu > \mu_0$ by constructing the test statistic

$$z = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

and comparing it to $z_{1-\alpha}$. In this case, with $\mu_0 = 70$ and $\alpha = 0.05$, we have

$$z = \frac{71.26 - 70}{11.48/\sqrt{200}} = \frac{1.26}{0.812} = 1.55,$$

which is less than $z_{0.95} = 1.645$. Therefore we do not reject $H_0 : \mu = 70$ in favor of $H_1 : \mu > 70$.

2c. The general formula for power in testing $H_0 : \mu = \mu_0$ against $H_1 : \mu > \mu_0$ at level α is

$$\Phi\left(-z_{1-\alpha} + \frac{|\mu_0 - \mu_1|\sqrt{n}}{\sigma}\right).$$

[Note that μ_1 must be larger than μ_0 to apply this formula, for otherwise $H_1 : \mu > \mu_0$ would not be true.] We have $\alpha = 0.05$, so that $-z_{1-\alpha} = -z_{0.95} = -1.645$. We also have $\mu_0 = 70$ and $n = 400$. Taking $\mu_1 = 71.26$ and $\sigma = 11.48$, as we have no compelling reason to do otherwise, we find that the power is

$$\Phi\left(-1.645 + \frac{|70 - 71.26|\sqrt{400}}{11.48}\right) = \Phi\,(0.550) = 70.9\%.$$

2d. The general formula for sample size in testing $H_0 : \mu = \mu_0$ against $H_1 : \mu > \mu_0$ at level α, when one desires power $1 - \beta$, is

$$n = \frac{\sigma^2(z_{1-\beta} + z_{1-\alpha})^2}{(\mu_0 - \mu_1)^2}.$$

[Note that μ_1 must be larger than μ_0 to apply this formula, for otherwise $H_1 : \mu > \mu_0$ would not be true.] With $1 - \beta = 0.90$ we have $z_{1-\beta} = z_{0.90} = 1.282$. So the required sample size is

$$\frac{11.48^2(1.282 + 1.645)^2}{(70 - 71.26)^2} \approx 712.$$

2e. We can easily calculate that $\hat{p} = 38/200 = 0.190$ and $n = 200$. The $100(1 - \alpha)\%$ "large sample" confidence interval for p is

$$\hat{p} \pm z_{1-\alpha/2}\sqrt{\hat{p}(1 - \hat{p})/n}.$$

Putting $\alpha = 0.05$ and noting that $z_{1-\alpha/2} = z_{0.975} = 1.960$, we obtain

$$0.190 \pm 0.054, \text{ which is } 0.136 \text{ to } 0.244.$$

Since $n\hat{p}(1-\hat{p}) = 30.78$ is well in excess of 10, we are comfortable regarding the sample size as large enough to use the above formula for the confidence interval.