STA 580 — Spring 2011 — Dr. Charnigo

Written Assignment 2 Solutions

1a. From Written Assignment 1 we know that $\bar{x} = 144.80$, $s = 14.93$, and $n = 25$. The $100(1-\alpha)\%$ "small sample" confidence interval for μ is

$$\bar{x} \pm t_{n-1,1-\alpha/2} s/\sqrt{n}.$$

Putting $\alpha = 0.05$ and noting that $t_{n-1,1-\alpha/2} = t_{24,0.975} = 2.064$, we obtain

$$144.80 \pm 6.16, \text{ which is } 138.64 \text{ to } 150.96.$$

We are assuming that the distribution of systolic blood pressure measurements is normal within the population of non-exercising adults.

1b. We conduct a level α "small sample" test of $H_0 : \mu = \mu_0$ against $H_1 : \mu > \mu_0$ by constructing the test statistic

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

and comparing it to $t_{n-1,1-\alpha}$. In this case, with $\mu_0 = 140$ and $\alpha = 0.05$, we have

$$t = \frac{144.80 - 140}{14.93/\sqrt{25}} = \frac{4.80}{2.986} = 1.608,$$

which is less than $t_{24,0.95} = 1.711$. Therefore we do not reject $H_0 : \mu = 140$ in favor of $H_1 : \mu > 140$. If H_1 were true and the normality assumption from part a were correct, then the proportion of hypertensive individuals within the population of non-exercising adults would be in excess of 0.50. Indeed, let X be the systolic blood pressure for a randomly selected person within the population of non-exercising adults. We have $0.50 = P(X > \mu) < P(X > 140)$, the equality arising from the symmetry of a normal distribution about its mean and the inequality arising from the supposition that $\mu > 140$. (You can draw a picture of a bell curve to visualize the inequality.)

1c. The general formula for (approximate) power in testing $H_0 : \mu = \mu_0$ against $H_1 : \mu > \mu_0$ at level α is

$$\Phi\left(-z_{1-\alpha} + \frac{|\mu_0 - \mu_1|\sqrt{n}}{\sigma}\right).$$

[Note that μ_1 must be larger than μ_0 to apply this formula, for otherwise $H_1 : \mu > \mu_0$ would not be true.] We have $\alpha = 0.05$, so that $-z_{1-\alpha} = -z_{0.95} = -1.645$. We also have $\mu_0 = 140$ and $n = 30$. Taking $\mu_1 = 144.80$ and $\sigma = 14.93$, as we have no compelling reason to do otherwise, we find that the power is

$$\Phi\left(-1.645 + \frac{|140 - 144.80|\sqrt{30}}{14.93}\right) = \Phi(0.116) = 54.6\%.$$

1d. The general formula for (approximate) sample size in testing $H_0 : \mu = \mu_0$ against $H_1 : \mu > \mu_0$ at level α, when one desires power $1 - \beta$, is

$$n = \frac{\sigma^2(z_{1-\beta} + z_{1-\alpha})^2}{(\mu_0 - \mu_1)^2}.$$

[Note that μ_1 must be larger than μ_0 to apply this formula, for otherwise $H_1 : \mu > \mu_0$ would not be true.] With $1 - \beta = 0.90$ we have $z_{1-\beta} = z_{0.90} = 1.282$. So the required sample size is

$$\frac{14.93^2(1.282 + 1.645)^2}{(140 - 144.80)^2} \approx 83.$$

1e. The $100(1 - \alpha)\%$ confidence interval for σ^2 is

$$\frac{(n-1)s^2}{\chi^2_{n-1,1-\alpha/2}} \text{ to } \frac{(n-1)s^2}{\chi^2_{n-1,\alpha/2}}.$$

Putting $\alpha = 0.05$ and noting that $\chi^2_{n-1,1-\alpha/2} = \chi^2_{24,0.975} = 39.36$, $\chi^2_{n-1,\alpha/2} = \chi^2_{24,0.025} = 12.40$, we obtain

$$\frac{(24)14.93^2}{39.36} \text{ to } \frac{(24)14.93^2}{12.40}, \text{ which is } 135.9 \text{ to } 431.4.$$

Taking square roots yields the 95% confidence interval of 11.66 to 20.77 for σ.

2a. We can easily calculate that $\bar{x} = 32.31$, $s = 6.13$, and $n = 200$. The $100(1 - \alpha)\%$ "large sample" confidence interval for μ is

$$\bar{x} \pm z_{1-\alpha/2} s/\sqrt{n}.$$

Putting $\alpha = 0.10$ and noting that $z_{1-\alpha/2} = z_{0.95} = 1.645$, we obtain

$$32.31 \pm 0.71, \text{ which is } 31.60 \text{ to } 33.02.$$

The width of the 90% confidence interval is $2z_{0.95}s/\sqrt{n}$. The width of the 95% confidence interval is $2z_{0.975}s/\sqrt{n}$. Since $z_{0.95} < z_{0.975}$ (as we can verify directly from Table 3, but this is also evident if you draw a picture of a bell curve), the width of the 90% confidence interval must be less than that of the 95% confidence interval. Intuitively, we need a wider interval to have a higher degree of confidence that we have captured μ.

2b. We conduct a level α "large sample" test of $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$ by constructing the test statistic

$$z = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

and comparing it to $z_{1-\alpha/2}$. In this case, with $\mu_0 = 25$ and $\alpha = 0.05$, we have

$$z = \frac{32.31 - 25}{6.13/\sqrt{200}} = \frac{7.31}{0.433} = 16.88,$$

which is far greater than $z_{0.975} = 1.96$. Therefore we reject $H_0 : \mu = 25$ in favor of $H_1 : \mu \neq 25$.

2c. The general formula for power in testing $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$ at level α is

$$\Phi\left(-z_{1-\alpha/2} + \frac{|\mu_0 - \mu_1|\sqrt{n}}{\sigma}\right).$$

We have $\alpha = 0.05$, so that $-z_{1-\alpha/2} = -z_{0.975} = -1.96$. We also have $\mu_0 = 25$ and $n = 100$. Taking $\mu_1 = 32.31$ and $\sigma = 6.13$, as we have no compelling reason to do otherwise, we find that the power is

$$\Phi\left(-1.96 + \frac{|25 - 32.31|\sqrt{100}}{6.13}\right) = \Phi\,(9.96) \approx 100\%.$$

2d. Power becomes larger as the sample size increases and smaller as the sample size decreases. Since a sample size of 100 provides well above 80% power, the sample size required for 80% power will be considerably less than 100. The general formula for sample size in testing $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$ at level α, when one desires power $1 - \beta$, is

$$n = \frac{\sigma^2(z_{1-\beta} + z_{1-\alpha/2})^2}{(\mu_0 - \mu_1)^2}.$$

With $1 - \beta = 0.80$ we have $z_{1-\beta} = z_{0.80} = 0.842$. So the required sample size is

$$\frac{6.13^2(0.842 + 1.96)^2}{(25 - 32.31)^2} \approx 6.$$

Remark. If a calculation produces such a small sample size in practice, then we may wish to add 2 (Charnigo's rule of thumb) to compensate for the fact that the actual critical value will arise from a T distribution (Table 5) rather than from the Z distribution (Table 3).

2e. We can easily calculate that $\hat{p} = 132/200 = 0.660$ and $n = 200$. The $100(1 - \alpha)\%$ "large sample" confidence interval for p is

$$\hat{p} \pm z_{1-\alpha/2}\sqrt{\hat{p}(1 - \hat{p})/n}.$$

Putting $\alpha = 0.05$ and noting that $z_{1-\alpha/2} = z_{0.975} = 1.960$, we obtain

$$0.660 \pm 0.066, \text{ which is } 0.594 \text{ to } 0.726.$$

Since $n\hat{p}(1-\hat{p}) = 44.88$ is well in excess of 10, we are comfortable regarding the sample size as large enough to use the above formula for the confidence interval.