STA 580 — Spring 2011 — Dr. Charnigo

Written Assignment 3 Solutions

1a. Using notation as in Lecture 7, the test statistic is

$$\chi^2 = \frac{(n_1 - 1)s_x^2}{25} = \frac{(73)27.96}{25} = 81.64.$$

The upper critical value is

$$\chi^2_{n_1-1,1-\alpha/2} = \chi^2_{73,0.975} = 98.52,$$

and the lower critical value is

$$\chi^2_{n_1-1,\alpha/2} = \chi^2_{73,0.025} = 51.26.$$

Since 81.64 is neither greater than 98.52 nor less than 51.26, we do not reject $H_0 : \sigma_1^2 = 25$ in favor of $H_1 : \sigma_1^2 \neq 25$.

1b. We have

$$s^2 = \frac{(n_1 - 1)s_x^2 + (n_2 - 1)s_y^2}{n_1 + n_2 - 2} = \frac{(73)27.96 + (125)36.25}{198} = 33.19.$$

The test statistic is

$$t = \frac{\bar{x} - \bar{y}}{\sqrt{s^2(1/n_1 + 1/n_2)}} = \frac{35.09 - 30.68}{\sqrt{33.19(1/74 + 1/126)}} = \frac{4.41}{0.844} = 5.23.$$

The (upper) critical value is

$$t_{n_1+n_2-2,1-\alpha/2} = t_{198,0.975} = 1.972.$$

Since $5.23 > 1.972$ we reject $H_0 : \mu_1 = \mu_2$ in favor of $H_1 : \mu_1 \neq \mu_2$.

1c. The 95% confidence interval for $\mu_1 - \mu_2$ assuming $\sigma_1^2 = \sigma_2^2$ is

$$\bar{x} - \bar{y} \pm t_{n_1+n_2-2,1-\alpha/2}\sqrt{s^2(1/n_1 + 1/n_2)},$$

which is

$$4.41 \pm 1.972(0.844)$$

or 2.75 to 6.07. Exclusion of 0 from the 95% confidence interval for $\mu_1 - \mu_2$ assuming $\sigma_1^2 = \sigma_2^2$ corresponds to rejection of $H_0 : \mu_1 - \mu_2 = 0$ in favor of $H_1 : \mu_1 - \mu_2 \neq 0$ at level $\alpha = 0.05$ assuming $\sigma_1^2 = \sigma_2^2$.

1d. We have

$$df = \frac{(s_x^2/n_1 + s_y^2/n_2)^2}{(s_x^2/n_1)^2/(n_1 - 1) + (s_y^2/n_2)^2/(n_2 - 1)} = \frac{(27.96/74 + 36.25/126)^2}{(27.96/74)^2/(73) + (36.25/126)^2/(125)} = \frac{0.443}{0.00262} = 169.$$

The test statistic is

$$t = \frac{\bar{x} - \bar{y}}{\sqrt{s_x^2/n_1 + s_y^2/n_2}} = \frac{35.09 - 30.68}{\sqrt{27.96/74 + 36.25/126}} = \frac{4.41}{0.816} = 5.40.$$

The (upper) critical value is

$$t_{df,1-\alpha/2} = t_{169,0.975} = 1.974.$$

Since $5.40 > 1.974$ we reject $H_0 : \mu_1 = \mu_2$ in favor of $H_1 : \mu_1 \neq \mu_2$.

1e. The 95% confidence interval for $\mu_1 - \mu_2$ assuming $\sigma_1^2 \neq \sigma_2^2$ is

$$\bar{x} - \bar{y} \pm t_{df,1-\alpha/2}\sqrt{s_x^2/n_1 + s_y^2/n_2},$$

which is

$$4.41 \pm 1.974(0.816)$$

or 2.80 to 6.02. Exclusion of 0 from the 95% confidence interval for $\mu_1 - \mu_2$ assuming $\sigma_1^2 \neq \sigma_2^2$ corresponds to rejection of $H_0 : \mu_1 - \mu_2 = 0$ in favor of $H_1 : \mu_1 - \mu_2 \neq 0$ at level $\alpha = 0.05$ assuming $\sigma_1^2 \neq \sigma_2^2$.

1f. The test statistic is

$$f = \frac{s_x^2}{s_y^2} = \frac{27.96}{36.25} = 0.771.$$

The upper critical value is

$$f_{n_1-1,n_2-1,1-\alpha/2} = f_{73,125,0.975} = 1.491,$$

while the lower critical value is

$$f_{n_1-1,n_2-1,\alpha/2} = f_{73,125,0.025} = 0.656.$$

(This lower critical value can also be written as $1/1.525 = 1/f_{125,73,0.975} = 1/f_{n_2-1,n_1-1,1-\alpha/2}$.) Since $0.656 < 0.771 < 1.491$ we accept $H_0 : \sigma_1^2 = \sigma_2^2$. Given that we have accepted $H_0 : \sigma_1^2 = \sigma_2^2$, most people would say that the test and confidence interval in parts b and c are more appropriate than the test and confidence interval in parts d and e.

1g. The statement is true. If $H_0 : \sigma_1^2 = 25$ were rejected in favor of $H_1 : \sigma_1^2 \neq 25$, we would have either $\chi^2 < \chi^2_{n-1,0.025}$ ("Case 1") or $\chi^2 > \chi^2_{n-1,0.975}$ ("Case 2"). Since $\chi^2_{n-1,0.025} < \chi^2_{n-1,0.05}$, in Case 1 we would also have $\chi^2 < \chi^2_{n-1,0.05}$ and thus rejection of $H_0 : \sigma_1^2 = 25$ in favor of $H_1 : \sigma_1^2 < 25$. Since $\chi^2_{n-1,0.975} > \chi^2_{n-1,0.95}$, in Case 2 we would also have $\chi^2 > \chi^2_{n-1,0.95}$ and thus rejection of $H_0 : \sigma_1^2 = 25$ in favor of $H_1 : \sigma_1^2 > 25$.

2a. Using notation as in Lecture 6, the test statistic is

$$z = \frac{\hat{p} - p_0}{\sqrt{p_0(1-p_0)/n}} = \frac{72/74 - 0.50}{\sqrt{0.50(0.50)/74}} = \frac{0.473}{0.0581} = 8.14.$$

The critical value is

$$z_{1-\alpha} = z_{0.95} = 1.645.$$

Since $8.14 > 1.645$ we reject $H_0 : p = 0.50$ in favor of $H_1 : p > 0.50$.

2b. The power with a sample size of 80 diabetics is

$$\Phi\left[\sqrt{\frac{p_0(1-p_0)}{p_1(1-p_1)}}\left(-z_{1-\alpha} + \frac{\sqrt{n}|p_0 - p_1|}{\sqrt{p_0(1-p_0)}}\right)\right] = \Phi\left[\sqrt{\frac{0.50(0.50)}{72/74(2/74)}}\left(-1.645 + \frac{\sqrt{80}|0.50 - 72/74|}{\sqrt{0.50(0.50)}}\right)\right] =$$

$$\Phi\left[3.083(6.816)\right] = \Phi[21.01] \approx 1.$$

2c. For 80% power the sample size (number of diabetics) is

$$\frac{p_0(1-p_0)\left(z_{1-\alpha}+z_{1-\beta}\sqrt{\frac{p_1(1-p_1)}{p_0(1-p_0)}}\right)^2}{(p_1-p_0)^2} = \frac{0.50(0.50)\left(1.645+0.842\sqrt{\frac{72/74(2/74)}{0.50(0.50)}}\right)^2}{(72/74-0.50)^2} \approx 5.$$

Remark. With a sample size of 5 we would not actually use the large-sample testing procedure described in Lecture 6 but rather (a one-sided version of) the exact testing procedure described in Equation 7.44 of the textbook. (You are not responsible for the exact testing procedure in STA 580.) In this scenario, the exact testing procedure would reject $H_0 : p = 0.50$ in favor of $H_1 : p > 0.50$ if and only if all 5 subjects in the sample had BMI above 25. If p really were 0.972, then the probability of all 5 subjects in the sample having BMI above 25 would be 0.872 — a bit more, in fact, than the advertised 80% power.