STA 623 — Fall 2009 — Dr. Charnigo

Final Examination

This non-collaborative take-home final examination may be submitted any time between 12 Noon on Monday 14 December and 2 p.m. on Thursday 17 December. By non-collaborative I mean that you are not permitted to discuss the examination with anyone other than me, until after the deadline for submission. The examination is to be submitted in hard copy, to me in person or under my office door (CPH 203-B).

[30] 1. Suppose that X is a random variable whose moment generating function $M_X(t)$ exists for all t in a neighborhood of 0 and has the form $1 + \sum_{j=1}^{K} c_j t^j / j!$ for some positive integer K.

[10] a. Find $E[X^K]$ and $E[X^{2K}]$.

[10] b. Find $Var[X^K]$. Argue that c_K must be 0.

[10] c. Since $c_K = 0$, we may replace $1 + \sum_{j=1}^{K} c_j t^j / j!$ by $1 + \sum_{j=1}^{K-1} c_j t^j / j!$. But the same argument shows that $c_{K-1} = 0$, and then we can argue that $c_{K-2} = 0$, etc. Thus we see that a moment generating function cannot be a nonconstant polynomial. But show that a moment generating function can be a constant polynomial by exhibiting a random variable X such that $M_X(t) = 1$ for all real t.

[30] 2. Let X and Y have joint probability density function $f_{X,Y}(x, y) := Cy \exp[-xy] 1_{\{x>1, y>1\}}$.

[10] a. What must C equal?

[10] b. Find the marginal probability density function of X.

[10] c. Find the conditional probability density function of X given that $Y = y$. Are X and Y independent?

[40] 3. Let X have probability density function $f_X(x) := \theta x^{\theta-1} 1_{\{0<x<1\}}$ for some $\theta \in (0, \infty)$.

[10] a. Directly calculate $E[X]$.

[10] b. Assuming that $E[-\log X]$ exists, employ Jensen's Inequality to relate $E[-\log X]$ to $E[X]$.

[10] c. Directly calculate $E[-\log X]$.

[10] d. By Taylor expansion we have $\log(\theta + 1) = \log(\theta) + \theta^{-1} - (1/2)\widetilde{\theta}^{-2}$, where $\widetilde{\theta} \in [\theta, \theta + 1]$. Use this fact, along with the results of parts a and c, to directly confirm Jensen's Inequality from part b.