STA 623 — Fall 2009 — Dr. Charnigo

Mock Comprehensive Examination

This examination is a strictly individual activity. Textbooks, notes, calculators, computers, and technology with Internet access are prohibited. Record that which you want graded in the blue book.

[30] 1. Respond to each item below.

[10] a. State the definition of a moment generating function for a random variable X.

[10] b. Let X be a normal random variable with mean $\mu \in \mathbb{R}$ and standard deviation $\sigma \in (0, \infty)$. Put $Y := \exp[X]$. What is the probability density function of Y?

[10] c. Let X_1 and X_2 be independent normal random variables with respective means $\mu_1, \mu_2 \in \mathbb{R}$ and standard deviations $\sigma_1, \sigma_2 \in (0, \infty)$. Put $Y_1 := \exp[X_1]$, $Y_2 := \exp[X_2]$, and $U := Y_1 Y_2$. What is the probability density function of U? (A potentially useful fact is that the moment generating function of a normal random variable with mean $\mu \in \mathbb{R}$ and standard deviation $\sigma \in (0, \infty)$ is $\exp[\mu t + \sigma^2 t^2/2]$.)

[30] 2. Respond to each item below.

[10] a. Let X have the offset geometric distribution (supported on the positive integers) with parameter $p \in (0, 1)$. What is the cumulative distribution function of X?

[10] b. With X as defined in part a, let $Y := 1_{\{X \in \{3,6,9,\ldots\}\}}$. What is the probability mass function of Y?

[10] c. With X and Y as defined in parts a and b, what is the conditional probability mass function of X given that $Y = 1$?

[40] 3. Respond to each item below.

[10] a. State Jensen's Inequality for a random variable X supported on $(0, \infty)$ and a real-valued convex function g defined on $(0, \infty)$. (You do not need to define convexity.)

[10] b. Let X have the inverse gamma distribution with shape parameter $\alpha \in (2, \infty)$ and scale parameter $\beta \in (0, \infty)$. The probability density function of X is $f_X(x) = \frac{1}{\Gamma[\alpha]\beta^\alpha} x^{-(\alpha+1)} \exp[-1/(\beta x)] 1_{\{x>0\}}$. Calculate $E[X]$. (As you can see, this distribution is closely related to the gamma distribution with shape parameter α and scale parameter β.)

[10] c. With X as defined in part b, put $Y := 1/X$. Taking for granted that both expectations exist, apply Jensen's Inequality to determine a relationship between $E[X]$ and $E[Y]$. Conclude that $Cov[X, Y] < 0$.

[10] d. With X and Y as defined in parts b and c, calculate $E[Y]$. Then calculate $Cov[X, Y]$ to verify your conclusion from part c.