

# STA 623 – Fall 2009 – Dr. Charnigo

## Section 1.1: Set Theory

*Sample space.* Suppose that we conduct an experiment. The set of all possible outcomes from that experiment is referred to as the sample space. When convenient, we use  $S$  as a symbol for the sample space.

**Example (sample space).** Suppose that I flip a coin four times. Critique the following statement: “The sample space is  $\{0, 1, 2, 3, 4\}$ , where the number refers to how many times the coin lands on heads.”

*Event.* Suppose that every element of a set  $A$  is an element of the sample space  $S$ , which we write as  $A \subset S$ . Then we refer to  $A$  as an event.

**Example (event).** Critique the following statement: “Events are elements of the sample space.”

*Union, intersection, and complementation operators.* Suppose that  $A, B \subset S$ . We define the union  $A \cup B$  as the set containing all elements of  $S$  present in  $A$  or  $B$  or both. We define the intersection  $A \cap B$  as the set containing all elements of  $S$  present in both  $A$  and  $B$ . We define the complementation  $A^c$  as the set containing all elements of  $S$  not present in  $A$ .

**Example (union, intersection, and complementation operators).** Suppose that I roll a six-sided die. The sample space is  $\{1,2,3,4,5,6\}$ , where the number refers to the result of the roll. Let  $A$  be the event that I roll an odd number and  $B$  be the event that I roll a multiple of 3. What are  $A \cup B$ ,  $A \cap B$ ,  $A^c$ ,  $A \cup A^c$ ,  $A \cap A^c$ , and  $(A \cup A^c)^c$ ?

*Properties of operators.* Suppose that  $A, B, C \subset S$ . We have the following.

1. Commutativity:  $A \cup B = B \cup A$ ,  $A \cap B = B \cap A$ .
2. Associativity:  $A \cup (B \cup C) = (A \cup B) \cup C$ ,  $A \cap (B \cap C) = (A \cap B) \cap C$ .
3. Distributivity:  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ ,  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ .
4. DeMorgan laws:  $(A \cup B)^c = A^c \cap B^c$ ,  $(A \cap B)^c = A^c \cup B^c$ .

**Example (properties of operators).** Critique the following attempt at proof of the first DeMorgan law: “Suppose that  $B = A^c$ . On the one hand,  $A \cup B = S$  and  $(A \cup B)^c = S^c = \emptyset$ . On the other hand,  $B^c = A$  and  $A^c \cap B^c = A^c \cap A = \emptyset$ . This completes the proof.”

*Proving that two sets are the same.* There is a standard way to prove that two sets are the same: show that each is a subset of the other. For such proofs the following notation is helpful:  $\in$  means “is an element of”, while  $\notin$  means “is not an element of”.

**Example (proving that two sets are the same).** Let us prove the first DeMorgan law. Suppose that  $x \in (A \cup B)^c$ . Then  $x \notin A \cup B$ . Hence  $x \notin A$  and  $x \notin B$ . So  $x \in A^c$  and  $x \in B^c$ . Hence  $x \in A^c \cap B^c$ . This shows that  $(A \cup B)^c \subset A^c \cap B^c$ . Starting from  $x \in A^c \cap B^c$  and working our way back to  $x \in (A \cup B)^c$  shows that  $A^c \cap B^c \subset (A \cup B)^c$ .

*General unions and intersections.* Suppose that  $A_\gamma \subset S$  for all  $\gamma \in \Gamma$ , where  $\Gamma$  is some non-empty index set. Then we define  $\cup_{\gamma \in \Gamma} A_\gamma$  to contain all elements of  $S$  present in at least one of the  $A_\gamma$  and  $\cap_{\gamma \in \Gamma} A_\gamma$  to contain all elements of  $S$  present in all of the  $A_\gamma$ . Common choices for  $\Gamma$  include  $\{1,2\}$ , which reproduces our earlier definitions, and  $\{1,2,3,\dots\}$ , which gives rise to so-called countable unions and intersections.

**Example (general unions and intersections).** Suppose that  $A_i = [0, i]$  for  $i \in \{1, 2, 3, \dots\}$ . What are  $\cup_{i=1}^\infty A_i$  and  $\cap_{i=1}^\infty A_i$ ? For generic  $A_1, A_2, A_3, \dots \subset S$ , show that  $\cap_{n=1}^\infty \cup_{i=n}^\infty A_i$  consists of all elements of  $S$  that belong to infinitely many of  $A_1, A_2, A_3, \dots$  and that  $\cup_{n=1}^\infty \cap_{i=n}^\infty A_i$  consists of all elements of  $S$  that belong to all but finitely many of  $A_1, A_2, A_3, \dots$ .

*Mutually exclusive and collectively exhaustive.* Suppose that  $A_\gamma \subset S$  for all  $\gamma \in \Gamma$ , where  $\Gamma$  is some non-empty index set. If  $A_i \cap A_j = \emptyset$  for all unequal  $i, j \in \Gamma$ , then we say that the  $A_\gamma$  are mutually exclusive. If  $\cup_{\gamma \in \Gamma} A_\gamma = S$ , then we say that the  $A_\gamma$  are collectively exhaustive. If the  $A_\gamma$  are both mutually exclusive and collectively exhaustive, then we say that they constitute a partition of  $S$ .