STA 623 – Fall 2009 – Dr. Charnigo

Section 1.4: Random Variables

Motivating illustration. Suppose that we conduct a taste test in which each of 10 grocery shoppers is asked to try "Brand A" and "Brand B" of peanut butter and then state which one he or she prefers. Assuming that a response of "no preference" is not allowed, the sample space S consists of 2^{10} elements.

Two distinct elements of S are $A, A, A, B, B, B, B, B, B, B$ and $B, B, B, B, B, B, B, A, A, A$. However, if all we want to know is how many people liked Brand A better versus how many people liked Brand B better, then these two elements carry the same information: 3 people liked Brand A better and 7 people liked Brand B better. In fact, we may consider defining X to be the number of people who like Brand A better and just reporting the observed value of X rather than the observed element of S.

Example (motivating illustration). Suppose that all elements of S in the motivating illustration above are equally likely. What is the probability that $X = 0$? What is the probability that $X = 1$? What is the probability that $X = 2$? The third axiom of probability gives us an interesting identity, $2^{10} =$.

Random variable. Suppose that the sample space S has an accompanying sigma field $\mathcal{B}$. Let the set of real numbers $\mathbb{R}$ also be endowed with a sigma field $\mathcal{B}^1$, specifically the smallest sigma field containing all open subintervals of $\mathbb{R}$. A function $X : S \to \mathbb{R}$ is called a random variable if, for every $B \in \mathcal{B}^1$, we have $\{\omega : X(\omega) \in B\} \in \mathcal{B}$. Here we are using ω to represent a generic element of S.

Example (random variable). In the motivating illustration above we have $X(A, A, A, A, A, A, B, B, B, B) =$. Since $\{\omega : X(\omega) \in B\} \subset S$ for any $B \in \mathcal{B}^1$, we are assured that X is a random variable if we take $\mathcal{B}$ to consist of all subsets of S.

If for some reason we had chosen $\mathcal{B}$ to consist only of S and $\emptyset$, then X would not be a random variable since, for example, $\{\omega : X(\omega) \in (9.5, 10.5)\} = \{A, A, A, A, A, A, A, A, A, A\} \notin \mathcal{B}$. Of course, such a choice of $\mathcal{B}$ is contrived and unnatural. In STA 623 we will not get into any trouble if we just regard a random variable as a function that maps elements of S to elements of $\mathbb{R}$.

Probabilities involving random variables. Let X be a random variable. For any $B \in \mathcal{B}^1$, we formally define $P(X \in B)$ as $P(\{\omega : X(\omega) \in B\})$. For $B, B_1, B_2, \ldots \in \mathcal{B}^1$, we have the following properties:

1. $P(X \in B) \geq 0$.
2. $P(X \in \mathbb{R}) = 1$.
3. $P(X \in \cup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} P(X \in B_i)$ when $B_i \cap B_j = \emptyset$ for $i \neq j$.

Example (probabilities involving random variables). Let us verify property 3 above. We have

$$\begin{aligned}
P(X \in \cup_{i=1}^{\infty} B_i) &= P(\{\omega : X(\omega) \in \cup_{i=1}^{\infty} B_i\}) \\
&= P(\cup_{i=1}^{\infty} \{\omega : X(\omega) \in B_i\}) \\
&= \sum_{i=1}^{\infty} P(\{\omega : X(\omega) \in B_i\}) = \sum_{i=1}^{\infty} P(X \in B_i).
\end{aligned}$$

To understand the second equality, let $A := \{\omega : X(\omega) \in \cup_{i=1}^{\infty} B_i\}$ and $A_i := \{\omega : X(\omega) \in B_i\}$. The second equality is just saying that

$P(A) = P(\cup_{i=1}^{\infty} A_i)$, which follows from the identity $A = \cup_{i=1}^{\infty} A_i$.

(If $\omega \in A$, then $X(\omega) \in \cup_{i=1}^{\infty} B_i$. So there must exist $i \in \{1, 2, \ldots\}$ such that $X(\omega) \in B_i$, which implies that $\omega \in A_i$ and hence $\omega \in \cup_{i=1}^{\infty} A_i$. This shows that $A \subset \cup_{i=1}^{\infty} A_i$. On the other hand, if $\omega \in \cup_{i=1}^{\infty} A_i$, then $\omega \in A_i$ for some $i \in \{1, 2, \ldots\}$. So we have $X(\omega) \in B_i$ and hence $X(\omega) \in \cup_{i=1}^{\infty} B_i$. This means that $\omega \in A$, so we have also shown that $\cup_{i=1}^{\infty} A_i \subset A$.)

The third equality is just saying that $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$. This is true because $A_i \cap A_j = \emptyset$ for $i \neq j$.

(If $\omega \in A_i \cap A_j$, then $X(\omega) \in B_i \cap B_j = \emptyset$, which contradicts the definition of a random variable since $X(\omega)$ should be a real number.)

Practical considerations. Suppose that we take systolic blood pressure measurements on 10 subjects and let X denote the average of the 10 measurements. We may wish to answer questions like, what is $P(140 \leq X \leq 150)$?

We can explicitly define S to be a subset of $\mathbb{R}^{10}$ — are there any obvious restrictions on S? — and put $X(\omega) := \sum_{i=1}^{10} \omega_i/10$, where ω is a vector whose first component ω_1 contains the measurement for the first subject, whose second component ω_2 contains the measurement for the second subject, and so forth.

Yet, the three properties above tell us, in effect, that we can perform computations involving random variables without explicitly defining S or a probability structure on its accompanying sigma field.

Example (practical considerations). Suppose that $P(X \geq 140) = 0.40$ and $P(X > 150) = 0.20$. What is $P(140 \leq X \leq 150)$?

Suppose that $P(X < x) = P(X \leq x) = 1 - \exp[-x]$ for any $x \in [0, \infty)$. Here we are using (capital) X to denote a random variable and (lower case) x to denote a possible observed value of X. What is $P(X > 1)$? What is $P(|X - 2| > 1)$?