

## STA 623 – Fall 2009 – Dr. Charnigo

### Section 2.1: Distributions of functions of a random variable

*Probabilities for functions of a random variable.* Let  $X$  be a random variable. Put  $Y := g(X)$  for some function  $g$  such that, for any set  $B \in \mathcal{B}^1$ , we have  $g^{-1}(B) := \{x \in \mathbb{R} : g(x) \in B\} \in \mathcal{B}^1$ . (This condition is satisfied for all  $g$  of practical interest.) Then, for any  $B \in \mathcal{B}^1$ , we define  $P(Y \in B) := P(X \in g^{-1}(B))$ .

**Example (probabilities for functions of a random variable).** Suppose that  $X$  has probability density function  $\exp[-2|x|]$ . Put  $Y := g(X) := X^2$  and  $B := [0, y]$  for a generic nonnegative real  $y$ . Then  $g^{-1}(B) =$   
and  $P(Y \leq y) =$

*Discrete and continuous cases.* If  $X$  is a discrete random variable, then so is  $Y$ . In this case, taking  $B := \{y\}$  for a generic  $y \in \mathbb{R}$  shows that

$$P(Y = y) = \sum_{x \in g^{-1}(\{y\}): P(X=x) > 0} P(X = x).$$

If  $X$  is a continuous random variable, the same may or may not be true of  $Y$ . For instance, let  $X$  have probability density function  $1_{\{x>0\}} \exp[-x]$ . Then  $Y := X^2$  is a continuous random variable,  $Y := \lfloor X \rfloor$  is a discrete random variable (here  $\lfloor X \rfloor$ , read “the floor of  $X$ ”, is the largest integer less than or equal to  $X$ ), and  $Y := \min\{X, 2\}$  is neither a discrete random variable nor a continuous random variable.

**Example (discrete and continuous cases).** Let  $X$  have probability mass function  $(1/2)^x$  for  $x \in \{1, 2, \dots\}$ . Let  $Y := \lfloor X \rfloor \bmod 2$ , so that  $Y = 1$  when  $\lfloor X \rfloor$  is odd and  $Y = 0$  otherwise. We have  $P(Y = 0) =$

*Monotonicity of the transforming function.* Let  $f_X(x)$  denote the probability mass (if  $X$  is a discrete random variable) or density (if  $X$  is a continuous random variable) function of  $X$ . Let  $\mathcal{X} := \{x \in \mathbb{R} : f_X(x) > 0\}$ , which we refer to as the support of  $X$ , and let  $\mathcal{Y} := \{y \in \mathbb{R} : y = g(x) \text{ for some } x \in \mathcal{X}\}$ .

First suppose that  $g(x)$  is strictly increasing, in that  $u > v$  implies  $g(u) > g(v)$ . Then, for any  $y \in \mathcal{Y}$  there exists a unique  $x \in \mathcal{X}$  such that  $g(x) = y$ . We refer to this  $x$  as  $g^{-1}(y)$ . Since  $g(X) \leq y$  is equivalent to  $X \leq g^{-1}(y)$ , we have

$$F_Y(y) = F_X(g^{-1}(y)),$$

where  $F_Y$  and  $F_X$  denote the cumulative distribution functions of  $Y$  and  $X$  respectively. In addition, if  $X$  is a continuous random variable,  $f_X(x)$  is continuous on  $\mathcal{X}$ , and  $g^{-1}(y)$  has a continuous derivative on  $\mathcal{Y}$ , then  $Y$  is a continuous random variable with probability density function

$$f_Y(y) = f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y) 1_{\{y \in \mathcal{Y}\}}.$$

Next suppose that  $g(x)$  is strictly decreasing, in that  $u > v$  implies  $g(u) < g(v)$ . Again, for any  $y \in \mathcal{Y}$  there exists a unique  $x \in \mathcal{X}$  such that  $g(x) = y$ . We refer to this  $x$  as  $g^{-1}(y)$ . Since  $g(X) \leq y$  is equivalent to  $X \geq g^{-1}(y)$ , we have

$$F_Y(y) = 1 - F_X(g^{-1}(y)) + P(X = g^{-1}(y)).$$

(Where does the last term above come from?) In addition, if  $X$  is a continuous random variable,  $f_X(x)$  is continuous on  $\mathcal{X}$ , and  $g^{-1}(y)$  has a continuous

derivative on  $\mathcal{Y}$ , then  $Y$  is a continuous random variable with probability density function

$$f_Y(y) = -f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y) 1_{\{y \in \mathcal{Y}\}}.$$

There is also a formula in your textbook for  $f_Y(y)$  when, among other conditions,  $\mathcal{X}$  can be partitioned into finitely many intervals on each of which  $g(x)$  is monotone (i.e., either strictly increasing or strictly decreasing). Personally I think that deriving  $F_Y(y)$  from first principles and then differentiating in  $y$  is easier than trying to remember the textbook formula.

**Example (monotonicity of the transforming function).** Suppose that  $X$  has probability density function  $\exp[-x]1_{\{x>0\}}$  with corresponding cumulative distribution function  $(1 - \exp[-x])1_{\{x>0\}}$ . Put  $g(x) := 1 - \exp[-x]$ , which is strictly increasing and which maps  $\mathcal{X} = (0, \infty)$  to  $\mathcal{Y} = (0, 1)$ . We have  $g^{-1}(y) = -\log(1 - y)$ , from which we find that a probability density function for  $Y$  is In  
 fact, this illustrates a general result called the probability integral transformation: whenever  $g(x) = F_X(x)$  for a continuous random variable  $X$ , we obtain this probability density function for  $Y$ . (The general result is more challenging to verify because there exist continuous random variables whose cumulative distribution functions are not strictly increasing.)

Suppose that  $X$  has probability density function  $\exp[-2|x|]$ . Put  $g(x) := x^2$ , which is clearly not monotone. However, since we found  $P(Y \leq y)$  earlier, we can differentiate in  $y$  to obtain an expression that is valid for all  $y \in (0, \infty)$ . We can define  $f_Y(y)$  to be zero for all  $y \in (-\infty, 0]$ , and then  $f_Y(y)$  will be a probability density function for  $Y$ .