STA 623 – Fall 2009 – Dr. Charnigo

Section 2.3: Moments and moment generating functions

Moments. Let X be a random variable. For any integer $n \geq 1$, we define the n^{th} moment of X to be $E[X^n]$. For any integer $n \geq 2$, we define the n^{th} central moment of X to be $E[(X-\nu)^n]$, where $\nu := E[X]$ is assumed to exist as a finite number. If the n^{th} moment exists as a finite number, then so do all moments of lower order. Hence, if the n^{th} moment exists as a finite number, then so does the n^{th} central moment.

Example (moments). Suppose X has probability density function

$$f_X(x) := (2\pi)^{-1/2}\sigma^{-1}\exp[-(x-\mu)^2/(2\sigma^2)],$$

where $\mu \in (-\infty,\infty)$ and $\sigma \in (0,\infty)$. Let $g(x) := (x-\mu)/\sigma$ and put $Z := g(X)$. Then for any real z we have

$$P(Z \leq z) = P(X \leq \sigma z + \mu) = \int_{-\infty}^{\sigma z+\mu} (2\pi)^{-1/2}\sigma^{-1}\exp[-(x-\mu)^2/(2\sigma^2)]\ dx.$$

Making the substitutions $y := (x-\mu)/\sigma$ and $dy := dx/\sigma$, we can express the integral as

$$\int_{-\infty}^{z} (2\pi)^{-1/2}\exp[-y^2/2]\ dy.$$

Upon differentiating with respect to z, we obtain a probability density function for Z of

$$f_Z(z) := (2\pi)^{-1/2}\exp[-z^2/2].$$

We have

$$E[Z^2] = (2\pi)^{-1/2}\int_{\mathbb{R}} z^2\exp[-z^2/2]\ dz = (2\pi)^{-1/2}\int_{\mathbb{R}} z\exp[-z^2/2]z\ dz.$$

Integrating by parts with $u := z$, $du := dz$, $dv := \exp[-z^2/2]z\ dz$, and $v := -\exp[-z^2/2]$, we obtain

$E[Z^2] =$

Likewise, for any positive integer p, we have

$$E[Z^{2p}] = (2\pi)^{-1/2}\int_{\mathbb{R}} z^{2p}\exp[-z^2/2]\ dz = (2\pi)^{-1/2}\int_{\mathbb{R}} z^{2p-1}\exp[-z^2/2]z\ dz.$$

Integrating by parts with $u := z^{2p-1}$, $du := (2p-1)z^{2p-2}\,dz$, $dv := \exp[-z^2/2]z\,dz$, and $v := -\exp[-z^2/2]$, we obtain

$$E[Z^{2p}] = (2\pi)^{-1/2}(2p-1)\int_{\mathbb{R}} z^{2p-2}\exp[-z^2/2]\ dz = (2p-1)E[Z^{2p-2}].$$

Thus, by mathematical induction,

$$E[Z^{2p}] = (2p-1)\times(2p-3)\times\cdots\times 3\times 1,$$

which some mathematicians denote $(2p-1)!!$, read "double factorial".

Note that $E[Z^{2p-1}]$ must exist as a finite number (because $E[Z^{2p}]$ does) and must equal zero (because $E[Z^{2p-1}]$ is the integral of an odd function over $\mathbb{R}$).

Now we are in a good position to find some moments of $X = \sigma Z + \mu$.

We have

$E[X] =$

$E[X^2] =$

$E[(X-\nu)^2] =$

$E[X^3] = E[(\sigma Z+\mu)^3] = \sigma^3 E[Z^3] + 3\sigma^2\mu E[Z^2] + 3\sigma\mu^2 E[Z] + \mu^3 = 3\sigma^2\mu + \mu^3,$

and

$$E[X^4] = E[(\sigma Z+\mu)^4] = \sigma^4 E[Z^4] + 6\sigma^2\mu^2 E[Z^2] + \mu^4 = 3\sigma^4 + 6\sigma^2\mu^2 + \mu^4.$$

We conclude this set of examples with a cautionary note. Suppose W has probability density function $f_W(w) := \pi^{-1}/(1+w^2)$. Then

$$E[W^2] = \pi^{-1}\int_{\mathbb{R}} w^2/(1+w^2)\,dw \geq \pi^{-1}\int_1^\infty w^2/(1+w^2)\,dw \geq \pi^{-1}\int_1^\infty 1/2\,dw = \infty.$$

On the other hand,

$$E[W^3] = \pi^{-1} \int_{\mathbb{R}} w^3/(1+w^2)\ dw$$

appears to be 0 because the integrand is an odd function. If that is true, then we are in trouble because we have said that the existence of $E[W^3]$ as a finite number should imply the same for $E[W^2]$. How can this apparent contradiction be resolved?

Variance and standard deviation. The second central moment $E[(X-\nu)^2]$ is called the variance of X. The variance describes how much X fluctuates around its expected value. The standard deviation of X is defined to be the positive square root of the variance. Unlike the variance, the standard deviation is expressed in the same units as X. For instance, if X represents systolic blood pressure in mmHg, then the standard deviation of X is expressed in mmHg while the variance is expressed in $(\text{mmHg})^2$.

Three useful results, assuming all expectations and variances referred to exist as finite numbers, are as follows.

1. For any constants a and b, $Var[aX+b] = a^2 Var[X]$.
2. A computational formula for the variance is $E[X^2] - (E[X])^2$.
3. If $Var[X] = 0$, then $P(X = E[X]) = P(|X - E[X]| = 0) = 1$.

A useful result on expectation of indicators, for our next example. For any random variable X and any set $A \in \mathcal{B}^1$ we have $E[1_{\{X \in A\}}] = P(X \in A)$. To see this, suppose for concreteness that X is continuous with probability density function $f_X(x)$. Then we have

$$E[1_{\{X\in A\}}] = \int_{\mathbb{R}} 1_{\{x \in A\}} f_X(x)\ dx = \int_A f_X(x)\ dx = P(X \in A).$$

Example (variance and standard deviation). Your textbook authors prove the first two useful results above by appealing to linearity of expectation. We can prove the third by contradiction. Indeed, suppose that $E[(X - E[X])^2] = 0$ but that $P(|X - E[X]| > 0) = \epsilon > 0$. Since

$$\{|X - E[X]| > 0\} = \cup_{j=1}^{\infty}\{1/j \leq |X - E[X]| < 1/(j-1)\}$$

we have

$$0 < \epsilon = \sum_{j=1}^{\infty} P(1/j \leq |X - E[X]| < 1/(j-1)).$$

By countable additivity, there must exist $j \in \{1, 2, \ldots\}$ such that

$$0 < \delta = P(1/j \leq |X - E[X]| < 1/(j-1)) \leq P(1/j \leq |X - E[X]|).$$

Then, using monotonicity of expectation (twice) and the useful result on expectation of indicators, we have

$$\begin{aligned} E[(X - E[X])^2] &\geq E[(X - E[X])^2 1_{\{|X - E[X]| \geq 1/j\}}] \\ &\geq E[(1/j)^2 1_{\{|X - E[X]| \geq 1/j\}}] \\ &\geq (1/j)^2 P(|X - E[X]| \geq 1/j) \\ &\geq \\ &> 0. \end{aligned}$$

We have arrived at a contradiction. Therefore, we must conclude that $E[(X - E[X])^2] = 0$ implies $P(|X - E[X]| > 0) = 0$.

Moment generating function. The moment generating function of X is defined as $M_X(t) := E[\exp(tX)]$. The moment generating function is potentially useful for three reasons.

1. Suppose there exists $h > 0$ such that $M_X(t) < \infty$ for all $t \in [-h, h]$. Then, for every positive integer n, $E[X^n]$ exists as a finite number and is equal to $\frac{d^n}{dt^n} M_X(t)|_{t=0}$.

2. Suppose there exists $h > 0$ such that $M_X(t), M_Y(t) < \infty$ for all $t \in [-h, h]$. If $M_X(t) = M_Y(t)$ for all $t \in [-h, h]$, then X and Y have the same cumulative distribution function: $F_X(u) = F_Y(u)$ for any real u.

3. Suppose there exists $h > 0$ such that $M_X(t), M_{X_1}(t), M_{X_2}(t), \ldots < \infty$ for all $t \in [-h, h]$. If $M_{X_i}(t) \stackrel{i \to \infty}{\to} M_X(t)$ for all $t \in [-h, h]$, then the cumulative distribution functions of $X_1, X_2, \ldots$ converge to the cumulative distribution function of X at all points where the latter is continuous: $F_{X_i}(u) \stackrel{i \to \infty}{\to} F_X(u)$ for any real u at which $F_X(u)$ is continuous.

Example (moment generating function). To see why the first result above holds, suppose for concreteness that X is continuous with probability density function $f_X(x)$. For $t \in (-h, h)$ we have

$$\begin{aligned} \frac{d^n}{dt^n} M_X(t) &= \frac{d^n}{dt^n} \int_{\mathbb{R}} \exp[tx] f_X(x)\, dx \\ &= \int_{\mathbb{R}} \frac{\partial^n}{\partial t^n} \exp[tx] f_X(x)\, dx \\ &= \int_{\mathbb{R}} x^n \exp[tx] f_X(x)\, dx. \end{aligned}$$

(The second equality above, interchange of differentiation and integration, will be justified next week.) Hence,

$$\frac{d^n}{dt^n} M_X(t)|_{t=0} = \int_{\mathbb{R}} x^n f_X(x)\, dx = E[X^n].$$

Suppose X has probability density function

$$f_X(x) := (2\pi)^{-1/2} \sigma^{-1} \exp[-(x-\mu)^2/(2\sigma^2)],$$

where $\mu \in (-\infty, \infty)$ and $\sigma \in (0, \infty)$. Put $Z := (X - \mu)/\sigma$. Then Z has probability density function

$$f_Z(z) := (2\pi)^{-1/2} \exp[-z^2/2],$$

from which we can calculate the moment generating function of Z:

$$\begin{aligned} M_Z(t) &= (2\pi)^{-1/2} \int_{\mathbb{R}} \exp[tz] \exp[-z^2/2]\ dz \\ &= (2\pi)^{-1/2} \int_{\mathbb{R}} \exp[(-z^2 + 2tz)/2]\ dz \\ &= (2\pi)^{-1/2} \int_{\mathbb{R}} \exp[(-z^2 + 2tz - t^2)/2] \exp[t^2/2]\ dz \\ &= (2\pi)^{-1/2} \exp[t^2/2] \int_{\mathbb{R}} \exp[-(z-t)^2/2]\ dz \\ &= \end{aligned}$$

We can then calculate the moment generating function of $X = \sigma Z + \mu$:

$$\begin{aligned} M_X(t) = E[\exp(tX)] = E[\exp(t\sigma Z + t\mu)] &= \exp(t\mu) E[\exp(t\sigma Z)] \\ = \exp(t\mu) M_Z(t\sigma) = \end{aligned}$$

To see an application of the third result above, suppose X_i has probability density function

$$\frac{1}{\Gamma[\alpha_i]\beta_i^{\alpha_i}} x^{\alpha_i - 1} \exp[-x/\beta_i] 1_{\{x>0\}}$$

for $i \in \{1, 2, \ldots\}$, where $\alpha_i := i$ and $\beta_i := 1/i$. One can show that

$$M_{X_i}(t) = \left[\frac{1}{1 - \beta_i t}\right]^{\alpha_i} = \left[\frac{1}{1 - t/i}\right]^i$$

for $t \in (-i, i)$. Taking this for granted, we have

$$M_{X_i}(t) = \left[1 + \frac{t/i}{1 - t/i}\right]^i \overset{i\to\infty}{\to} \exp[t].$$

On the other hand, $\exp[t]$ is clearly $M_X(t)$, where X is a random variable that equals 1 with probability 1. As such, we conclude that for any $u \neq 1$

$$P(X_i \le u) \overset{i\to\infty}{\to} P(X \le u).$$

In particular,

$$P(X_i \le u) \overset{i\to\infty}{\to} 0 \ \text{ for } \ u < 1 \ \text{ and}$$
$$P(X_i \le u) \overset{i\to\infty}{\to} 1 \ \text{ for } \ u > 1.$$