STA 623 — Fall 2009 — Dr. Charnigo

Written Assignment 1 Solutions

1. If $x \in (A \cap B)^c$, then $x \notin A \cap B$. So either $x \notin A$ or $x \notin B$. Hence $x \in A^c$ or $x \in B^c$, which shows that $x \in A^c \cup B^c$. Thus $(A \cap B)^c \subset A^c \cup B^c$.

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2. If $x \in \cup_{n=1}^{\infty} \cap_{i=n}^{\infty} A_i$, then for some n we have $x \in \cap_{i=n}^{\infty} A_i$. This means that $x \in A_n$, $x \in A_{n+1}$, $x \in A_{n+2}$, and so forth. Clearly x is present in all but finitely many of $A_1, A_2, A_3, \ldots$. Thus $\cup_{n=1}^{\infty} \cap_{i=n}^{\infty} A_i \subset B$.

If x is present in all but finitely many of $A_1, A_2, A_3, \ldots$, then for some n we have $x \in A_n$, $x \in A_{n+1}$, $x \in A_{n+2}$, and so forth. Hence $x \in \cap_{i=n}^{\infty} A_i$, which shows that $x \in \cup_{n=1}^{\infty} \cap_{i=n}^{\infty} A_i$. Thus $B \subset \cup_{n=1}^{\infty} \cap_{i=n}^{\infty} A_i$.

3. The sports writer's statements translate to the assertions that $P(\{A\}) = 1/3$, $P(\{B\}) = 1/4$, $P(\{C\}) = 1/5$, $P(\{D\}) = 1/6$, $P(\{E\}) = 1/7$, and $P(\{F\}) = 1/8$. Here we use the obvious notation that, for example, $P(\{A\})$ is the probability that Horse A wins. Since only one Horse can win, the axioms of probability suggest that

$$
\begin{aligned}
1 &\geq P(\{A\} \cup \{B\} \cup \{C\} \cup \{D\} \cup \{E\} \cup \{F\}) \\
&= P(\{A\}) + P(\{B\}) + P(\{C\}) + P(\{D\}) + P(\{E\}) + P(\{F\}) \\
&> 1/4 + 1/4 + 1/8 + 1/8 + 1/8 + 1/8 = 1.
\end{aligned}
$$

But 1 cannot be strictly greater than itself, so the sports writer's statements are not compatible with the axioms of probability.

4. We have $1 \geq P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$, whence

$$P(A_1 \cap A_2) \geq P(A_1) + P(A_2) - 1. \tag{1}$$

Since $A_1 \cap A_2 \subset A_1$ we have $P(A_1 \cap A_2) \leq P(A_1)$, and since $A_1 \cap A_2 \subset A_2$ we have $P(A_1 \cap A_2) \leq P(A_2)$, so that $P(A_1 \cap A_2) \leq \min\{P(A_1), P(A_2)\}$.

In anticipation of a mathematical induction argument, suppose that

$$\sum_{i=1}^{k} P(A_i) - k + 1 \leq P(\cap_{i=1}^{k} A_i) \tag{2}$$

for $k = (n-1)$. The statement (2) holds for $k = 2$ because of (1). Then

$$P(\cap_{i=1}^{n} A_i) = P(\cap_{i=1}^{k} A_i \cap A_n) \geq P(\cap_{i=1}^{k} A_i) + P(A_n) - 1 \geq \sum_{i=1}^{k} P(A_i) + P(A_n) - k + 1 - 1 = \sum_{i=1}^{n} P(A_i) - n + 1, \tag{3}$$

where the first inequality in (3) follows from (1) and the second inequality follows from (2). This shows that (2) holds for $k = n$, completing the mathematical induction argument. Since $\cap_{i=1}^{n} A_i \subset A_j$ for every $j \in \{1, \ldots, n\}$, we have $P(\cap_{i=1}^{n} A_i) \leq P(A_j)$ for every $j \in \{1, \ldots, n\}$, whence $P(\cap_{i=1}^{n} A_i) \leq \min\{P(A_1), \ldots, P(A_n)\}$.

5. The argument presented is flawed because the 6 possibilities mentioned in the argument are not equally likely. Indeed, suppose that the people calling me are named "A" and "B". Then, taking into account

which call comes from which person, there are 9 equally likely possibilities: A and B both call on Monday, A calls on Monday and B calls on Tuesday, A calls on Monday and B calls on Wednesday, A calls on Tuesday and B calls on Monday, A and B both call on Tuesday, A calls on Tuesday and B calls on Wednesday, A calls on Wednesday and B calls on Monday, A calls on Wednesday and B calls on Tuesday, A and B both call on Wednesday. Thus, for example, the probability of getting one call on Monday and another Tuesday is twice the probability of getting both calls on Monday. In particular, we see that 5 of the 9 equally likely possibilities entail at least one call on Monday, so that the requested probability is 5/9.

6. The total number of 10-card gin hands is $\binom{52}{10} = 15820024220$. The total number of 10-card gin hands with two pairs and two triples is $\binom{13}{2}\binom{11}{2}\binom{4}{2}^2\binom{4}{3}^2 = 2471040$. Above, $\binom{13}{2}$ is the number of ways to choose the denominations for the two pairs, $\binom{11}{2}$ is the number of ways to choose the denominations for the two triples, $\binom{4}{2}$ is the number of ways to choose the suits for a pair, and $\binom{4}{3}$ is the number of ways to choose the suits for a triple. [We have $\binom{13}{2}$, not 13×12, for the number of ways to choose the denominations for the two pairs because, for example, a pair of 5's and a pair of 8's is indistinguishable from a pair of 8's and a pair of 5's.] Thus, the probability of receiving a 10-card gin hand with two pairs and two triples is

$$\frac{2471040}{15820024220} \approx 0.0156\%.$$

7. The total number of 6-ball drawings is $\binom{44}{6} = 7059052$. There is only one 6-ball drawing with which my ticket matches on all 6 numbers. There are $\binom{6}{5}\binom{38}{1} = 228$ 6-ball drawings with which my ticket matches on 5 numbers, and there are $\binom{6}{4}\binom{38}{2} = 10545$ 6-ball drawings with which my ticket matches on 4 numbers. Hence, the probability that I will win a prize is

$$\frac{1 + 228 + 10545}{7059052} \approx 0.153\%.$$

8. Let A denote the event that a person has HIV and B the event that a person tests positive for HIV. We have $P(A) = 0.006$, $P(B|A) = 0.95$, and $P(B^c|A^c) = 0.95$. From these facts we also know that $P(A^c) = 0.994$ and $P(B|A^c) = 0.05$. We want to find $P(A|B)$. An application of Bayes' Theorem yields

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)} = \frac{0.0057}{0.0057 + 0.0497} \approx 10.3\%.$$

9. In the first scenario, we have

$$P(A \cap B) = P(\{6\}) = 1/6 = 1/2 \times 1/3 = P(A)P(B),$$

so that A and B are independent. In the second scenario we have

$$P(A \cap B) = P(\{6\}) = 2/9 \neq 5/9 \times 1/3 = P(A)P(B),$$

so that A and B are not independent. We do not need $P(\{1\}) = P(\{2\}) = P(\{3\}) = P(\{4\}) = P(\{5\}) = P(\{6\}) = 1/6$ for independence of A and B. In fact, a sufficient condition is $P(\{1, 5\}) = P(\{2, 4\}) = 2P(\{3\}) = 2P(\{6\}) = 1/3$ since then

$$P(A \cap B) = P(\{6\}) = 1/6 = 1/2 \times 1/3 = P(A)P(B).$$

10. If A and B are mutually exclusive, then $A \cap B = \emptyset$ so that $P(A \cap B) = 0$. If A and B are also independent, then $0 = P(A \cap B) = P(A)P(B)$, so that either $P(A) = 0$ or $P(B) = 0$. Hence, a necessary condition for two events to be both mutually exclusive and independent is that at least one of the events have zero probability. [If all nonempty subsets of S have positive probability, then a necessary and sufficient condition for two events to be both mutually exclusive and independent is that at least one of the events be the empty set.]