STA 623 — Fall 2009 — Dr. Charnigo

Written Assignment 2 Solutions

1a. We have

$$\{X = 0\} = (E_1^c \cap E_2^c \cap E_3^c)$$

and so

$$P(X = 0) = P(E_1^c)P(E_2^c)P(E_3^c) = (1-p)^3.$$

1b. We have

$$\{X = 1\} = (E_1^c \cap E_2^c \cap E_3) \cup (E_1^c \cap E_2 \cap E_3^c) \cup (E_1 \cap E_2^c \cap E_3^c)$$

and so

$$P(X = 1) = P(E_1^c)P(E_2^c)P(E_3) + P(E_1^c)P(E_2)P(E_3^c) + P(E_1)P(E_2^c)P(E_3^c) = 3p(1-p)^2.$$

1c. We have

$$\{X = 2\} = (E_1^c \cap E_2 \cap E_3) \cup (E_1 \cap E_2^c \cap E_3) \cup (E_1 \cap E_2 \cap E_3^c)$$

and so

$$P(X = 2) = P(E_1^c)P(E_2)P(E_3) + P(E_1)P(E_2^c)P(E_3) + P(E_1)P(E_2)P(E_3^c) = 3p^2(1-p).$$

1d. We have

$$\{X = 3\} = (E_1 \cap E_2 \cap E_3)$$

and so

$$P(X = 3) = P(E_1)P(E_2)P(E_3) = p^3.$$

1e. The cumulative distribution function of X is

$$P(X \leq x) = (1-p)^3 1_{\{x \geq 0\}} + 3p(1-p)^2 1_{\{x \geq 1\}} + 3p^2(1-p) 1_{\{x \geq 2\}} + p^3 1_{\{x \geq 3\}}.$$

1f. For X and Y to be identically distributed we must have

$$P(X = 0) = P(Y = 3) = P(Y = 0) \quad \text{and} \quad P(X = 1) = P(Y = 2) = P(Y = 1).$$

This happens if and only if $p = 0.5$.

1g. We have

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \quad \text{for } x \in \{0, 1, \ldots, n\}.$$

Justification: This is obvious if $x \in \{0, n\}$. If $x \in \{1, \ldots, n-1\}$, then the probability that the first x shoppers prefer Brand A and the last $(n-x)$ shoppers prefer Brand B is $p^x(1-p)^{n-x}$. However, this is not the only way in which we can have $X = x$. Indeed, there are $\binom{n}{x}$ ways to have x shoppers prefer Brand A and $(n-x)$ shoppers prefer Brand B, each of which has the same probability. So the probability that x shoppers prefer Brand A and $(n-x)$ shoppers prefer Brand B is $\binom{n}{x}p^x(1-p)^{n-x}$.

1h. For a fixed $x \in \{0, 1, \ldots, n\}$, we want to maximize $\binom{n}{x}p^x(1-p)^{n-x}$ with respect to $p \in [0, 1]$, which is the same as maximizing $p^x(1-p)^{n-x}$. If $x = 0$, then this amounts to maximizing $(1-p)^n$, and we obviously have $\widehat{p} = 0$. If $x = n$, then this amounts to maximizing p^n, and we obviously have $\widehat{p} = 1$. If $x \in \{1, \ldots, n-1\}$, then clearly $p^x(1-p)^{n-x}$ is not maximized at 0 or 1, so the maximum must occur on the interior $(0, 1)$. Furthermore, the maximizer of $p^x(1-p)^{n-x}$ on $(0, 1)$ coincides with the maximizer of $x \log p + (n-x)\log(1-p)$ on $(0, 1)$, for which the calculations are slightly easier. The first derivative of $x \log p + (n-x)\log(1-p)$ on $(0, 1)$ is $x/p - (n-x)/(1-p)$. Setting the first derivative to zero yields $\widehat{p} = x/n$. We see readily that $x/p - (n-x)/(1-p) > 0$ for $p \in (0, \widehat{p})$ and $x/p - (n-x)/(1-p) < 0$ for $p \in (\widehat{p}, 1)$, establishing that $\widehat{p} = x/n$ is indeed the maximizer of $x \log p + (n-x)\log(1-p)$ on $(0, 1)$ and therefore the maximizer of $p^x(1-p)^{n-x}$ on $[0, 1]$. In fact, the formula $\widehat{p} = x/n$ is valid for $x \in \{0, n\}$ as well as for $x \in \{1, \ldots, n-1\}$.

2a. We must have

$$\int_0^1 Cx(1-x)\, dx = C\int_0^1 (x - x^2)\, dx = C(1/2 - 1/3) = C(1/6) = 1,$$

whence $C = 6$.

2b. The cumulative distribution function of X is

$$F_X(x) = \int_0^x 6t(1-t)\, dt = 6\int_0^x (t - t^2)\, dt = 6(x^2/2 - x^3/3) \quad \text{for } x \in [0, 1].$$

We have $F_X(x) = 0$ for $x < 0$ and $F_X(x) = 1$ for $x > 1$.

2c. Let y be a positive real. We have

$$P(Y \leq y) = P(X \geq \exp[-y]) = P(X > \exp[-y]) = 1 - F_X(\exp[-y]) = 1 - 6(\exp[-2y]/2 - \exp[-3y]/3).$$

We have $P(Y \leq y) = 0$ for $y \leq 0$.

2d. We have

$$\frac{d}{dy}P(Y \leq y) = 6(\exp[-2y] - \exp[-3y]) \quad \text{for } y > 0.$$

As such, we may take

$$6(\exp[-2y] - \exp[-3y])1_{\{y>0\}}$$

as a probability density function of Y.