# STA 623 — Fall 2009 — Dr. Charnigo

## Written Assignment 3 Solutions

1a. We take the parameter space for $p$ to be $(0,1)$. If $p = 1$, we will never achieve a failure and so $P(X \in \mathbb{R}) = 0$, contradicting the definition of a random variable. If $p = 0$, we will never achieve a success and so $P(X \in \mathbb{R}) = 0$, again contradicting the definition of a random variable.

1b. Note that at least two trials are required. For any $x \in \{2, 3, \ldots\}$ we have

$$P(X = x) = P(x-1 \text{ failures then success}) + P(x-1 \text{ successes then failure}) = (1-p)^{x-1}p + p^{x-1}(1-p).$$

To verify that this defines a probability mass function, we compute

$$\sum_{x=2}^{\infty}\{(1-p)^{x-1}p + p^{x-1}(1-p)\} = \sum_{u=1}^{\infty}(1-p)^u p + \sum_{u=1}^{\infty} p^u(1-p) = \frac{(1-p)p}{1-(1-p)} + \frac{p(1-p)}{1-p} = 1.$$

2a. We know that

$$\sum_{x=1}^{\infty} xq^{x-1} = \frac{1}{(1-q)^2}$$

for $q \in (0,1)$. We would like to differentiate in $q$ to conclude that

$$\sum_{x=2}^{\infty} x(x-1)q^{x-2} = \frac{2}{(1-q)^3}$$

for $q \in (0,1)$. To this end, let $y$ be a fixed element of $(0,1)$, let $\delta := \min\{y/2, (1-y)/2\}$, and put $g(x,q) := xq^{x-1}$ for $x \in \{1, 2, \ldots\}$ and $q \in (0,1)$.

1. Since $(y-\delta, y+\delta) \subset (0,1)$, we have

$$\sum_{x=1}^{\infty} g(x,q) = \sum_{x=1}^{\infty} xq^{x-1} < \infty$$

for $q \in (y-\delta, y+\delta)$.

2. For any fixed $x \in \{1, 2, \ldots\}$, we have

$$\frac{\partial}{\partial q} g(x,q) = \frac{\partial}{\partial q} xq^{x-1} = x(x-1)q^{x-2} = x(x-1)\exp[(x-2)\log q],$$

which is obviously continuous in $q \in (y-\delta, y+\delta)$.

3. For $q \in (y-\delta, y+\delta)$ and $x \in \{1, 2, \ldots\}$, we have

$$x(x-1)\exp[(x-2)\log q] \le x(x-1)\exp[(x-2)\log(y+\delta)].$$

There exists a positive integer $C$ such that $x(x-1)\exp[(x-2)\log(y+\delta)]$ is strictly decreasing as a function of $x \in [C, \infty)$. Hence,

$$\sum_{x=C+1}^{\infty} x(x-1)\exp[(x-2)\log(y+\delta)] \le \int_C^{\infty} x(x-1)\exp[(x-2)\log(y+\delta)]\ dx,$$

and the latter is obviously finite. Also $\sum_{x=1}^{C} x(x-1)\exp[(x-2)\log(y+\delta)]$ is clearly finite, whence

$$\sum_{x=1}^{\infty} x(x-1)\exp[(x-2)\log(y+\delta)] = \sum_{x=1}^{C} x(x-1)\exp[(x-2)\log(y+\delta)] + \sum_{x=C+1}^{\infty} x(x-1)\exp[(x-2)\log(y+\delta)] < \infty.$$

Putting $h(x) := x(x-1)\exp[(x-2)\log(y+\delta)]$, we have $0 \le \frac{\partial}{\partial q} g(x,q) \le h(x)$ for $q \in (y-\delta, y+\delta)$ and $\sum_{x=1}^{\infty} h(x) < \infty$.

Points 1, 2, and 3 imply that $\sum_{x=1}^{\infty} x(x-1)y^{x-2} = \frac{2}{(1-y)^3}$. But since this argument can be repeated with $y$ any element of $(0,1)$, we have $\sum_{x=1}^{\infty} x(x-1)q^{x-2} = \frac{2}{(1-q)^3}$ for $q \in (0,1)$. Also, the first term in $\sum_{x=1}^{\infty} x(x-1)q^{x-2}$ is zero, so we may as well write $\sum_{x=2}^{\infty} x(x-1)q^{x-2}$. Finally, if $q = 0$ then with the convention $0^0 := 1$ we have $\sum_{x=1}^{\infty} x0^{x-1} = 1 = \frac{1}{(1-0)^2}$ and $\sum_{x=2}^{\infty} x(x-1)0^{x-2} = 2 = \frac{2}{(1-0)^3}$.

2b. Recalling that $q = 1-p$ and that the probability mass function of $X$ is $pq^{x-1}$ for $x \in \{1, 2, \ldots\}$, we have

$$E[X] = \sum_{x=1}^{\infty} xpq^{x-1} = p\sum_{x=1}^{\infty} xq^{x-1} = p/(1-q)^2 = 1/p,$$

$$E[X(X-1)] = \sum_{x=2}^{\infty} x(x-1)pq^{x-1} = pq\sum_{x=2}^{\infty} x(x-1)q^{x-2} = 2pq/(1-q)^3 = 2q/p^2,$$

$$E[X^2] = E[X(X-1)] + E[X] = 1/p + 2q/p^2,$$

and

$$Var[X] = E[X^2] - (E[X])^2 = 1/p + 2q/p^2 - 1/p^2 = q/p^2.$$

3a. Let $X_t$ denote the number of phone messages received by time $t$. If the first phone message has not been received by time $t$, then the number of phone messages received by time $t$ is 0. That is, $\{T > t\} \subset \{X_t = 0\}$. Likewise, if the number of phone messages received by time $t$ is 0, then the first phone message has not been received by time $t$. That is, $\{X_t = 0\} \subset \{T > t\}$. So

$$P(T > t) = P(X_t = 0) = \exp[-\lambda t](\lambda t)^0/0! = \exp[-\lambda t].$$

Hence, $T$ has the exponential distribution with rate parameter $\lambda$. (If you do not recognize this survival function, then obtain and recognize the cumulative distribution function or the probability density function of $T$.)

3b. We have $\{V > t\} = \{T > t\} \cap \{U > t\}$ since the first message of either kind arrives after time $t$ if and only if both the first phone message and the first text message arrive after time $t$. Mimicking the strategy used for part a yields $P(U > t) = \exp[-\mu t]$, and by our independence assumption we have $P(V > t) = P(T > t)P(U > t) = \exp[-(\lambda + \mu)t]$. Thus, $V$ has the exponential distribution with rate parameter $\lambda + \mu$. As a check of this answer, let $Y_t$ denote the total number of messages received by time $t$. Then $Y_t$ has the Poisson distribution with mean $(\lambda + \mu)t$, and mimicking the strategy used for part a yields $P(V > t) = P(Y_t = 0) = \exp[-(\lambda + \mu)t]$.

4a. We have

$$M_{Z_n}(t) = E[\exp(tZ_n)] = E[\exp(tn^{-1/2}\{X_n - n\})]$$

$$= E[\exp(tn^{-1/2}X_n)]\exp(-tn^{1/2}) = M_{X_n}(tn^{-1/2})\exp(-tn^{1/2}) = \exp[n(\exp[tn^{-1/2}] - 1)]\exp(-tn^{1/2}).$$

4b. For any sufficiently differentiable function $g(x)$, we have

$$g(u) = g(0) + ug'(0) + u^2 g''(0)/2! + u^3 g'''(\xi_u)/3!$$

for some $\xi_u$ between 0 and $u$. In particular,

$$\exp[u] = 1 + u + u^2/2! + u^3 \exp[\xi_u]/3!.$$

Substituting $tn^{-1/2}$ for $u$, we have

$$\exp[tn^{-1/2}] - 1 = tn^{-1/2} + t^2 n^{-1}/2 + t^3 \exp[\xi_u] n^{-3/2}/6,$$

with $\exp[\xi_u] \le \exp[|t|n^{-1/2}]$. Then

$$M_{Z_n}(t) = \exp[n(\exp[tn^{-1/2}] - 1)] \exp(-tn^{1/2}) = \exp[n\{tn^{-1/2} + t^2 n^{-1}/2 + t^3 \exp[\xi_u] n^{-3/2}/6\} - tn^{1/2}]$$
$$= \exp[t^2/2] \exp[t^3 \exp[\xi_u] n^{-1/2}/6].$$

Since $0 \le |t^3 \exp[\xi_u] n^{-1/2}/6| = |t|^3 \exp[\xi_u] n^{-1/2}/6 \le |t|^3 \exp[|t|n^{-1/2}] n^{-1/2}/6 \to 0$, we have $M_{Z_n}(t) \to \exp[t^2/2]$, the moment generating function of the standard normal distribution. This suggests that, for large $n$, $Z_n$ can be treated as if it were approximately standard normal and $X_n = n^{1/2} Z_n + n$ can be treated as if it were approximately normal with mean $n$ and variance $n$.

5a. The survival function is, for $x > 0$,

$$S(x) := P(X > x) = \int_x^\infty \beta/(t+1)^{\beta+1} \, dt = 1/(x+1)^\beta.$$

The hazard function is, for $x > 0$,

$$H(x) := -\frac{d}{dx} \log S(x) = -\frac{d}{dx}\{-\beta \log(x+1)\} = \beta/(x+1).$$

5b. Since the hazard function is strictly decreasing over $x \in (0, \infty)$, using $X$ as a model of a lifetime for a human being or a mechanical object seems inappropriate. We do not anticipate that younger people or objects have a greater propensity to expire than older people or objects. A phenomenon for which $X$ may be an appropriate model is the length of time between two successive occasions on which a person consumes an alcoholic drink. For example, a person who last drank two weeks ago seems more likely to drink today than does a person who last drank two years ago. Another phenomenon for which $X$ may be an appropriate model is the length of time for a web page to finish loading. For example, a web page seems more likely to finish loading in the next five seconds if it just started loading than if it has been loading for thirty seconds.