STA 623 — Fall 2009 — Dr. Charnigo

Written Assignment 3 Solutions

1a. We take the parameter space for p to be $(0,1)$. If $p = 1$, we will never achieve a failure and so $P(X \in \mathbb{R}) = 0$, contradicting the definition of a random variable. If $p = 0$, we will never achieve a success and so $P(X \in \mathbb{R}) = 0$, again contradicting the definition of a random variable.

1b. Note that at least two trials are required. For any $x \in \{2, 3, \ldots\}$ we have

$P(X = x) = P(x-1 \text{ failures then success}) + P(x-1 \text{ successes then failure}) = (1-p)^{x-1}p + p^{x-1}(1-p)$.

To verify that this defines a probability mass function, we compute

$$\sum_{x=2}^{\infty}\{(1-p)^{x-1}p + p^{x-1}(1-p)\} = \sum_{u=1}^{\infty}(1-p)^u p + \sum_{u=1}^{\infty} p^u(1-p) = \frac{(1-p)p}{1-(1-p)} + \frac{p(1-p)}{1-p} = 1.$$

2a. We know that

$$\sum_{x=1}^{\infty} x q^{x-1} = \frac{1}{(1-q)^2}$$

for $q \in (0,1)$. We would like to differentiate in q to conclude that

$$\sum_{x=2}^{\infty} x(x-1)q^{x-2} = \frac{2}{(1-q)^3}$$

for $q \in (0,1)$. To this end, let y be a fixed element of $(0,1)$, let $\delta := \min\{y/2, (1-y)/2\}$, and put $g(x,q) := xq^{x-1}$ for $x \in \{1, 2, \ldots\}$ and $q \in (0,1)$.

1. Since $(y-\delta, y+\delta) \subset (0,1)$, we have

$$\sum_{x=1}^{\infty} g(x,q) = \sum_{x=1}^{\infty} xq^{x-1} < \infty$$

for $q \in (y-\delta, y+\delta)$.

2. For any fixed $x \in \{1, 2, \ldots\}$, we have

$$\frac{\partial}{\partial q} g(x,q) = \frac{\partial}{\partial q} xq^{x-1} = x(x-1)q^{x-2} = x(x-1)\exp[(x-2)\log q],$$

which is obviously continuous in $q \in (y-\delta, y+\delta)$.

3. For $q \in (y-\delta, y+\delta)$ and $x \in \{1, 2, \ldots\}$, we have

$$x(x-1)\exp[(x-2)\log q] \le x(x-1)\exp[(x-2)\log(y+\delta)].$$

There exists a positive integer C such that $x(x-1)\exp[(x-2)\log(y+\delta)]$ is strictly decreasing as a function of $x \in [C, \infty)$. Hence,

$$\sum_{x=C+1}^{\infty} x(x-1)\exp[(x-2)\log(y+\delta)] \le \int_C^{\infty} x(x-1)\exp[(x-2)\log(y+\delta)]\ dx,$$

and the latter is obviously finite. Also $\sum_{x=1}^{C} x(x-1)\exp[(x-2)\log(y+\delta)]$ is clearly finite, whence

$$\sum_{x=1}^{\infty} x(x-1)\exp[(x-2)\log(y+\delta)] = \sum_{x=1}^{C} x(x-1)\exp[(x-2)\log(y+\delta)] + \sum_{x=C+1}^{\infty} x(x-1)\exp[(x-2)\log(y+\delta)] < \infty.$$

Putting $h(x) := x(x-1)\exp[(x-2)\log(y+\delta)]$, we have $0 \le \frac{\partial}{\partial q} g(x,q) \le h(x)$ for $q \in (y-\delta, y+\delta)$ and $\sum_{x=1}^{\infty} h(x) < \infty$.

Points 1, 2, and 3 imply that $\sum_{x=1}^{\infty} x(x-1)y^{x-2} = \frac{2}{(1-y)^3}$. But since this argument can be repeated with y any element of $(0,1)$, we have $\sum_{x=1}^{\infty} x(x-1)q^{x-2} = \frac{2}{(1-q)^3}$ for $q \in (0,1)$. Also, the first term in $\sum_{x=1}^{\infty} x(x-1)q^{x-2}$ is zero, so we may as well write $\sum_{x=2}^{\infty} x(x-1)q^{x-2}$. Finally, if $q = 0$ then with the convention $0^0 := 1$ we have $\sum_{x=1}^{\infty} x0^{x-1} = 1 = \frac{1}{(1-0)^2}$ and $\sum_{x=2}^{\infty} x(x-1)0^{x-2} = 2 = \frac{2}{(1-0)^3}$.

2b. Recalling that $q = 1-p$ and that the probability mass function of X is pq^{x-1} for $x \in \{1, 2, \ldots\}$, we have

$$E[X] = \sum_{x=1}^{\infty} xpq^{x-1} = p\sum_{x=1}^{\infty} xq^{x-1} = p/(1-q)^2 = 1/p,$$

$$E[X(X-1)] = \sum_{x=2}^{\infty} x(x-1)pq^{x-1} = pq\sum_{x=2}^{\infty} x(x-1)q^{x-2} = 2pq/(1-q)^3 = 2q/p^2,$$

$$E[X^2] = E[X(X-1)] + E[X] = 1/p + 2q/p^2,$$

and

$$Var[X] = E[X^2] - (E[X])^2 = 1/p + 2q/p^2 - 1/p^2 = q/p^2.$$

3a. Let X_t denote the number of phone messages received by time t. If the first phone message has not been received by time t, then the number of phone messages received by time t is 0. That is, $\{T > t\} \subset \{X_t = 0\}$. Likewise, if the number of phone messages received by time t is 0, then the first phone message has not been received by time t. That is, $\{X_t = 0\} \subset \{T > t\}$. So

$$P(T > t) = P(X_t = 0) = \exp[-\lambda t](\lambda t)^0/0! = \exp[-\lambda t].$$

Hence, T has the exponential distribution with rate parameter λ. (If you do not recognize this survival function, then obtain and recognize the cumulative distribution function or the probability density function of T.)

3b. We have $\{V > t\} = \{T > t\} \cap \{U > t\}$ since the first message of either kind arrives after time t if and only if both the first phone message and the first text message arrive after time t. Mimicking the strategy used for part a yields $P(U > t) = \exp[-\mu t]$, and by our independence assumption we have $P(V > t) = P(T > t)P(U > t) = \exp[-(\lambda+\mu)t]$. Thus, V has the exponential distribution with rate parameter $\lambda + \mu$. As a check of this answer, let Y_t denote the total number of messages received by time t. Then Y_t has the Poisson distribution with mean $(\lambda+\mu)t$, and mimicking the strategy used for part a yields $P(V > t) = P(Y_t = 0) = \exp[-(\lambda+\mu)t]$.

4a. We have

$$M_{Z_n}(t) = E[\exp(tZ_n)] = E[\exp(tn^{-1/2}\{X_n - n\})]$$
$$= E[\exp(tn^{-1/2}X_n)]\exp(-tn^{1/2}) = M_{X_n}(tn^{-1/2})\exp(-tn^{1/2}) = \exp[n(\exp[tn^{-1/2}] - 1)]\exp(-tn^{1/2}).$$

4b. For any sufficiently differentiable function $g(x)$, we have

$$g(u) = g(0) + ug'(0) + u^2 g''(0)/2! + u^3 g'''(\xi_u)/3!$$

for some ξ_u between 0 and u. In particular,

$$\exp[u] = 1 + u + u^2/2! + u^3 \exp[\xi_u]/3!.$$

Substituting $tn^{-1/2}$ for u, we have

$$\exp[tn^{-1/2}] - 1 = tn^{-1/2} + t^2 n^{-1}/2 + t^3 \exp[\xi_u] n^{-3/2}/6,$$

with $\exp[\xi_u] \leq \exp[|t|n^{-1/2}]$. Then

$$\begin{aligned} M_{Z_n}(t) = \exp[n(\exp[tn^{-1/2}] - 1)] \exp(-tn^{1/2}) &= \exp[n\{tn^{-1/2} + t^2 n^{-1}/2 + t^3 \exp[\xi_u] n^{-3/2}/6\} - tn^{1/2}] \\ &= \exp[t^2/2] \exp[t^3 \exp[\xi_u] n^{-1/2}/6]. \end{aligned}$$

Since $0 \leq |t^3 \exp[\xi_u] n^{-1/2}/6| = |t|^3 \exp[\xi_u] n^{-1/2}/6 \leq |t|^3 \exp[|t|n^{-1/2}] n^{-1/2}/6 \to 0$, we have $M_{Z_n}(t) \to \exp[t^2/2]$, the moment generating function of the standard normal distribution. This suggests that, for large n, Z_n can be treated as if it were approximately standard normal and $X_n = n^{1/2} Z_n + n$ can be treated as if it were approximately normal with mean n and variance n.

5a. The survival function is, for $x > 0$,

$$S(x) := P(X > x) = \int_x^\infty \beta/(t+1)^{\beta+1}\, dt = 1/(x+1)^\beta.$$

The hazard function is, for $x > 0$,

$$H(x) := -\frac{d}{dx} \log S(x) = -\frac{d}{dx}\{-\beta \log(x+1)\} = \beta/(x+1).$$

5b. Since the hazard function is strictly decreasing over $x \in (0, \infty)$, using X as a model of a lifetime for a human being or a mechanical object seems inappropriate. We do not anticipate that younger people or objects have a greater propensity to expire than older people or objects. A phenomenon for which X may be an appropriate model is the length of time between two successive occasions on which a person consumes an alcoholic drink. For example, a person who last drank two weeks ago seems more likely to drink today than does a person who last drank two years ago. Another phenomenon for which X may be an appropriate model is the length of time for a web page to finish loading. For example, a web page seems more likely to finish loading in the next five seconds if it just started loading than if it has been loading for thirty seconds.