STA 623 — Fall 2009 — Dr. Charnigo

Written Assignment 4

Written Assignment 4 is due on Thursday 05 November at the end of class. You are encouraged to work in groups of two or three, but you may work individually if you prefer.

[30] 1. Let X have the logistic distribution with probability density function $f(x) := \exp[-x]/(1+\exp[-x])^2$.

[10] a. Find the mean and the median of the distribution of X. [Hint: To find the mean, write $\int_{-\infty}^{\infty} xf(x)\ dx = \int_{-\infty}^{0} xf(x)\ dx + \int_{0}^{\infty} xf(x)\ dx$ and integrate by parts, noting that $\lim_{x\to\infty} x[F(x)-1] = \lim_{x\to-\infty} xF(x) = 0$ and that $1/(1+\exp[-x]) = 1 - \{\exp[-x]/(1+\exp[-x])\}$.]

[10] b. Use $f(x)$ to define a location-scale family of distributions. Explicitly specify both the functional form of $f(x;\mu,\sigma)$ and the parameter space.

[10] c. Find the mean and the median of each distribution in the location-scale family from part b.

[40] 2. Let X have the beta distribution with parameters $\alpha \in (1,\infty)$ and $\beta \in (0,\infty)$. Let $g(x)$ be continuously differentiable on $(0,1)$ with $\lim_{x\to 0} g(x)x^{\alpha-1} = \lim_{x\to 1} g(x)(1-x)^{\beta} = 0$.

[10] a. Show that $E[g(X)\{\beta - (\alpha-1)(1-X)/X\}] = E[(1-X)g'(X)]$, assuming that the latter exists as a finite number. [If the latter exists as a finite number, then so does the former.]

[10] b. Use your answer to part a, along with the result of exercise 3b from the take-home midterm examination, to find $E[\log X(2-X^{-1})]$ when $\beta = \alpha - 1 = C$, a positive integer.

[10] c. Show that the distribution of X belongs to an exponential family.

[10] d. Use your answer to part c to find $E[\log X]$ when $\beta = \alpha - 1 = C$. Then find $E[X^{-1}\log X]$ when $\beta = \alpha - 1 = C$. [Hint: For $x > 0$ we have $\psi(x+1) = \psi(x) + 1/x$, where $\psi(x) := \frac{d}{dx}\log\Gamma[x]$ is the digamma function.]

[30] 3. Pursue the following explorations of Chebychev's Inequality.

[10] a. Exhibit a random variable X with $E[X] = 0$ and $Var[X] = 1$ such that $P(|X - EX| \geq SD[X]) = P(|X| \geq 1) = 1$. [Hint: First find a nonnegative random variable Y such that $E[Y] = 1$ and $P(\sqrt{Y} \geq 1) = 1$.]

[10] b. Let $c \in (1,\infty)$ be fixed. Exhibit a random variable X with $E[X] = 0$ and $Var[X] = 1$ such that $P(|X - EX| \geq c\ SD[X]) = P(|X| \geq c) = 1/c^2$.

[10] c. Let X be a continuous random variable with $E[X] = 0$, $Var[X] = 1$, and probability density function $f_X(x)$. Show that $\lim_{c\to\infty} P(|X - E[X]| \geq c\ SD[X])/(1/c^2) = \lim_{c\to\infty} c^2 P(|X| \geq c) = 0$. [Hint: Put $Y := |X|$. Argue that $0 \leq \int_c^{\infty} c^2 f_Y(y)\ dy \leq \int_c^{\infty} y^2 f_Y(y)\ dy \to 0$ as $c \to \infty$.]