STA 623 — Fall 2009 — Dr. Charnigo

Written Assignment 5

Written Assignment 5 is due on Thursday 19 November at the end of class. You are encouraged to work in groups of two or three, but you may work individually if you prefer.

[30] 1. Use moment generating functions to find the distribution of $X+Y$, X and Y independent, in each of the following situations. You may take for granted that the moment generating functions are as reported in the back of your textbook; you do not need to derive them from scratch.

[10] a. X binomial with parameters $p \in [0,1]$ and $n_1 \in \{1,2,\ldots\}$, Y binomial with parameters $p \in [0,1]$ and $n_2 \in \{1,2,\ldots\}$

[10] b. X geometric with parameter $p \in (0,1]$, Y geometric with parameter $p \in (0,1]$

[10] c. X gamma with shape parameter $\alpha_1 \in (0,\infty)$ and scale parameter $\beta \in (0,\infty)$, Y gamma with shape parameter $\alpha_2 \in (0,\infty)$ and scale parameter $\beta \in (0,\infty)$

[40] 2. Suppose that X has the chi-square distribution on $p_1 \in (0,\infty)$ degrees of freedom and that, independently, Y has the chi-square distribution on $p_2 \in (0,p_1)$ degrees of freedom.

[10] a. Apply exercise 1c to find the distribution of $X+Y$.

[10] b. A naive guess might be that the distribution of $X-Y$ is chi-square on $p_1 - p_2$ degrees of freedom. Prove that such a guess is wrong by demonstrating that $P(X-Y<0)>0$. Hint: Show that $P(X<1, Y>1)>0$.

[10] c. Use the one-to-one bivariate transformation formula to find the joint distribution of $U := X+Y$ and $V := X-Y$.

[10] d. In the special case that $p_1 = 4$ and $p_2 = 2$, find the marginal distribution of $X-Y$.

[30] 3. Suppose that g is a nonnegative continuous function, not identically zero, on $[0,1/2] \subset \mathbb{R}$, and let g be defined arbitrarily outside of $[0,1/2]$. Put $\mathcal{A} := \{(x,y)' \in \mathbb{R}^2 : x>0, y>0, x^2+y^2<1\}$ and $h(x,y) := g(xy)1_{\{(x,y)' \in \mathcal{A}\}}$ for $(x,y)' \in \mathbb{R}^2$.

[10] a. Show that $\int_{\mathbb{R}^2} h(x,y)\ dx\ dy =: C$ is a finite positive number. Hint: A continuous function defined on a compact subinterval of $\mathbb{R}$ assumes a finite minimum value, call it a, and a finite maximum value, call it b. Moreover, the continuity of g on $[0,1/2]$ implies that h is continuous on $\mathcal{A}$. As such, there exists a ball $\mathcal{B}$ of positive radius ϵ, contained entirely in $\mathcal{A}$, such that $h(x,y) \geq b/2 > 0$ for $(x,y)' \in \mathcal{B}$. Argue that $0 < \int_{\mathcal{B}} b/2\ dx\ dy \leq \int_{\mathbb{R}^2} h(x,y)\ dx\ dy \leq \int_{\mathcal{A}} b\ dx\ dy < \infty$.

[10] b. Show that $f_{X,Y}(x,y) := C^{-1}h(x,y)$ defines a valid joint probability density function.

[10] c. For X and Y governed by $f_{X,Y}(x,y)$, calculate $P(X<Y)$. Hint: Write out the double integral for $P(X<Y)$ and see what happens when you replace the dummy variables of integration x and y by $\widetilde{y}$ and $\widetilde{x}$.