STA 623 — Fall 2009 — Dr. Charnigo

Written Assignment 5 Solutions

1a. Let $n := n_1 + n_2$. We have

$$M_{X+Y}(t) = M_X(t)M_Y(t) = [pe^t + (1-p)]^{n_1}[pe^t + (1-p)]^{n_2} = [pe^t + (1-p)]^n,$$

which is the moment generating function of a binomial random variable with parameters p and n.

1b. If we agree that there is no "offset" (i.e., the support set is taken to be the nonnegative integers), then

$$M_{X+Y}(t) = M_X(t)M_Y(t) = \frac{p}{1-(1-p)e^t}\frac{p}{1-(1-p)e^t} = \left(\frac{p}{1-(1-p)e^t}\right)^2 \quad \text{for } t < -\log(1-p).$$

If we agree that there is an offset, then

$$M_{X+Y}(t) = M_X(t)M_Y(t) = \frac{pe^t}{1-(1-p)e^t}\frac{pe^t}{1-(1-p)e^t} = \left(\frac{pe^t}{1-(1-p)e^t}\right)^2 \quad \text{for } t < -\log(1-p).$$

In the former case, we have the moment generating function of a negative binomial random variable with no offset and parameters 2 and p. In the latter case, we have the moment generating function of a negative binomial random variable with an offset and parameters 2 and p.

1c. Let $\alpha := \alpha_1 + \alpha_2$. We have

$$M_{X+Y}(t) = M_X(t)M_Y(t) = \left(\frac{1}{1-\beta t}\right)^{\alpha_1}\left(\frac{1}{1-\beta t}\right)^{\alpha_2} = \left(\frac{1}{1-\beta t}\right)^{\alpha} \quad \text{for } t < 1/\beta,$$

which is the moment generating function of a gamma random variable with parameters α and β.

2a. Put $\alpha_1 := p_1/2$, $\alpha_2 := p_2/2$, and $\beta := 2$. Since X has the gamma distribution with parameters α_1 and β, and since Y has the gamma distribution with parameters α_2 and β, we may apply exercise 1c to find that $X+Y$ has the gamma distribution with parameters $\alpha := \alpha_1 + \alpha_2 = (p_1+p_2)/2$ and β. Thus, $X+Y$ has the chi-square distribution on $p_1 + p_2$ degrees of freedom.

2b. Since $\{X < 1, Y > 1\} \subset \{X - Y < 0\}$, showing that $P(X < 1, Y > 1) > 0$ is enough to prove that $P(X - Y < 0) > 0$. Noting the independence of X and Y, we have

$$\begin{aligned} P(X < 1, Y > 1) &= P(X<1)P(Y>1) \\ &= \int_0^1 \frac{1}{\Gamma[p_1/2]2^{p_1/2}} x^{p_1/2-1}\exp[-x/2]\,dx \int_1^\infty \frac{1}{\Gamma[p_2/2]2^{p_2/2}} y^{p_2/2-1}\exp[-y/2]\,dy. \end{aligned}$$

If you have studied measure theory, you know that the integral of a nonnegative function over an interval of nonzero length can only be zero if the function is zero almost everywhere with respect to Lebesgue measure on that interval. Hence, both of the above integrals are strictly positive, showing that $P(X < 1, Y > 1)$ is strictly positive.

If you have not studied measure theory, note that $g_p(x) := x^{p/2-1}\exp[-x/2]1_{\{x>0\}}$ is strictly decreasing for $x \in (0,\infty)$ if $p_1 \in (0,2]$ and that the function is increasing for $x \in (0, p-2]$ and decreasing for $x \in [p-2,\infty)$ if $p \in (2,\infty)$. In the former case, note that the minimum of $g_p(x)$ over $x \in (0,1]$ is $g_p(1)$

and so

$$
\begin{aligned}
P(X<1) &= \int_0^1 \frac{1}{\Gamma[p_1/2]2^{p_1/2}} g_{p_1}(x)\ dx \\
&= \frac{1}{\Gamma[p_1/2]2^{p_1/2}} \int_0^1 g_{p_1}(x)\ dx \\
&\geq \frac{1}{\Gamma[p_1/2]2^{p_1/2}} \int_0^1 g_{p_1}(1)\ dx \\
&= \frac{g_{p_1}(1)}{\Gamma[p_1/2]2^{p_1/2}} \\
&> 0.
\end{aligned}
$$

In the latter case, note that the minimum of $g_p(x)$ over $x \in [1/2, 1]$ is either $g_p(1/2)$ or $g_p(1)$ and so

$$
\begin{aligned}
P(X<1) &= \int_0^1 \frac{1}{\Gamma[p_1/2]2^{p_1/2}} g_{p_1}(x)\ dx \\
&= \frac{1}{\Gamma[p_1/2]2^{p_1/2}} \int_0^1 g_{p_1}(x)\ dx \\
&\geq \frac{1}{\Gamma[p_1/2]2^{p_1/2}} \int_{1/2}^1 g_{p_1}(x)\ dx \\
&\geq \frac{1}{\Gamma[p_1/2]2^{p_1/2}} \int_{1/2}^1 \min\{g_{p_1}(1/2), g_{p_1}(1)\}\ dx \\
&= \frac{\min\{g_{p_1}(1/2), g_{p_1}(1)\}}{\Gamma[p_1/2]2^{p_1/2+1}} \\
&> 0.
\end{aligned}
$$

Likewise, since the minimum of $g_p(x)$ over $x \in (p, p+1]$ is $g_p(p+1)$, we have

$$
\begin{aligned}
P(Y>1) &= \int_1^\infty \frac{1}{\Gamma[p_2/2]2^{p_2/2}} g_{p_2}(x)\ dx \\
&\geq \int_{p_2}^{p_2+1} \frac{1}{\Gamma[p_2/2]2^{p_2/2}} g_{p_2}(x)\ dx \\
&= \frac{1}{\Gamma[p_2/2]2^{p_2/2}} \int_{p_2}^{p_2+1} g_{p_2}(x)\ dx \\
&\geq \frac{1}{\Gamma[p_2/2]2^{p_2/2}} \int_{p_2}^{p_2+1} g_{p_2}(p_2+1)\ dx \\
&= \frac{g_{p_2}(p_2+1)}{\Gamma[p_2/2]2^{p_2/2}} \\
&> 0.
\end{aligned}
$$

In summary, both $P(X<1)$ and $P(Y>1)$ are strictly positive, so their product $P(X<1, Y>1)$ is also strictly positive.

2c. Since $0 \leq |v| < \max\{x, y\} < u$, we have

$$S_{U,V} = \{(u,v)' \in \mathbb{R}^2 : u > 0, |v| < u\}.$$

We have $x = (u+v)/2 =: h_1(u,v)$ and $y = (u-v)/2 =: h_2(u,v)$, so that $\frac{\partial h_1(u,v)}{\partial u} = \frac{\partial h_1(u,v)}{\partial v} = \frac{\partial h_2(u,v)}{\partial u} = 1/2$ and $\frac{\partial h_2(u,v)}{\partial v} = -1/2$, yielding a matrix determinant with absolute value $1/2$. Recalling the independence of X and Y, we have

$$\begin{aligned}
f_{X,Y}(h_1(u,v), h_2(u,v)) &= f_X(h_1(u,v)) f_Y(h_2(u,v)) \\
&= \frac{1}{\Gamma[p_1/2]2^{p_1/2}} \left(\frac{u+v}{2}\right)^{p_1/2-1} \exp[-(u+v)/4] \frac{1}{\Gamma[p_2/2]2^{p_2/2}} \left(\frac{u-v}{2}\right)^{p_2/2-1} \exp[-(u-v)/4] \\
&= \frac{1}{\Gamma[p_1/2]\Gamma[p_2/2]2^{p_1+p_2-2}} (u+v)^{p_1/2-1}(u-v)^{p_2/2-1} \exp[-u/2]
\end{aligned}$$

and hence

$$f_{U,V}(u,v) = \frac{1}{\Gamma[p_1/2]\Gamma[p_2/2]2^{p_1+p_2-1}} (u+v)^{p_1/2-1}(u-v)^{p_2/2-1} \exp[-u/2]$$

for $u > 0, |v| < u$.

2d. If $p_1 = 4$ and $p_2 = 2$, then

$$f_{U,V}(u,v) = \frac{1}{32}(u+v)\exp[-u/2]$$

for $u > 0, |v| < u$. If $v \geq 0$, then since $u > v$ we have

$$\begin{aligned}
f_V(v) &= \int_v^\infty f_{U,V}(u,v)\ du \\
&= \int_v^\infty \frac{1}{32}(u+v)\exp[-u/2]\ du \\
&= \frac{1}{32}\int_v^\infty u\exp[-u/2]\ du + \frac{v}{32}\int_v^\infty \exp[-u/2]\ du \\
&= -\frac{1}{32}\exp[-u/2](2u+4)|_v^\infty - \frac{v}{32}\ 2\exp[-u/2]|_v^\infty \\
&= \frac{(v+1)}{8}\exp[-v/2].
\end{aligned}$$

If $v < 0$, then since $u > -v$ we have

$$\begin{aligned}
f_V(v) &= \int_{-v}^\infty f_{U,V}(u,v)\ du \\
&= -\frac{1}{32}\exp[-u/2](2u+4)|_{-v}^\infty - \frac{v}{32}\ 2\exp[-u/2]|_{-v}^\infty \\
&= \frac{1}{8}\exp[v/2].
\end{aligned}$$

So, a single formula that covers all $v \in \mathbb{R}$ is

$$f_V(v) = \frac{1}{8}\left(v1_{\{v>0\}} + 1\right)\exp[-|v|/2].$$

To verify that $f_V(v)$ is indeed a probability density function (a good way to check one's work in a computation-heavy exercise like this), note that

$$\begin{aligned}
\int_{-\infty}^{\infty} f_V(v)\ dv &= \int_{-\infty}^{0} f_V(v)\ dv + \int_{0}^{\infty} f_V(v)\ dv \\
&= \int_{-\infty}^{0} \frac{1}{8}\exp[v/2]\ dv + \int_{0}^{\infty} \frac{(v+1)}{8}\exp[-v/2]\ dv \\
&= \int_{0}^{\infty} \frac{1}{8}\exp[-w]\ 2\ dw + \int_{0}^{\infty} \frac{2w+1}{8}\exp[-w]\ 2\ dw \\
&= \int_{0}^{\infty} \frac{2}{8}\exp[-w]\ dw + \int_{0}^{\infty} \frac{4}{8} w\exp[-w]\ dw + \int_{0}^{\infty} \frac{2}{8}\exp[-w]\ dw \\
&= \frac{2}{8}\Gamma[1] + \frac{4}{8}\Gamma[2] + \frac{2}{8}\Gamma[1] \\
&= 1.
\end{aligned}$$

3a. We need to establish the four inequalities in "$0 < \int_{\mathcal{B}} b/2\ dx\ dy \leq \int_{\mathbb{R}^2} h(x,y)\ dx\ dy \leq \int_{\mathcal{A}} b\ dx\ dy < \infty$."

FIRST INEQUALITY. Since g is nonnegative continuous but not identically zero on $[0, 1/2]$, we must have $b > 0$. Moreover, we have $\{xy : (x,y)' \in \mathcal{A}\} = (0, 1/2)$. Therefore, the ball $\mathcal{B}$ described in the hint exists and

$$\int_{\mathcal{B}} b/2\ dx\ dy = (b/2)\int_{\mathcal{B}} dx\ dy = (b/2) \text{ Area of } \mathcal{B} = (b/2)\pi\epsilon^2 > 0.$$

SECOND INEQUALITY. Since h is nonnegative and $\mathcal{B} \subset \mathbb{R}^2$, we have

$$\int_{\mathcal{B}} b/2\ dx\ dy \leq \int_{\mathcal{B}} h(x,y)\ dx\ dy \leq \int_{\mathbb{R}^2} h(x,y)\ dx\ dy.$$

THIRD INEQUALITY. Since h is zero outside $\mathcal{A}$ and bounded above by the maximum of g over $[0, 1/2]$ inside $\mathcal{A}$, we have

$$\int_{\mathbb{R}^2} h(x,y)\ dx\ dy = \int_{\mathcal{A}} h(x,y)\ dx\ dy \leq \int_{\mathcal{A}} b\ dx\ dy.$$

FOURTH INEQUALITY. We have

$$\int_{\mathcal{A}} b\ dx\ dy = b\int_{\mathcal{A}} dx\ dy = b \text{ Area of } \mathcal{A} = b\pi/4 < \infty.$$

3b. Since $C \in (0, \infty)$ and $h(x,y) \geq 0$, we know that $f_{X,Y}(x,y) \geq 0$. Moreover, we have

$$\int_{\mathbb{R}^2} f_{X,Y}(x,y)\ dx\ dy = \int_{\mathbb{R}^2} C^{-1}h(x,y)\ dx\ dy = C^{-1}\int_{\mathbb{R}^2} h(x,y)\ dx\ dy = C^{-1}C = 1.$$

Therefore, $f_{X,Y}(x,y)$ defines a valid joint probability density function.

3c. We have (drawing a picture will help)

$$\begin{aligned}
P(X < Y) &= \int_{0}^{1/\sqrt{2}} \int_{x}^{\sqrt{1-x^2}} f_{X,Y}(x,y)\ dy\ dx \\
&= \int_{0}^{1/\sqrt{2}} \int_{x}^{\sqrt{1-x^2}} C^{-1}g(xy)\ dy\ dx \\
&= \int_{0}^{1/\sqrt{2}} \int_{\widetilde{y}}^{\sqrt{1-\widetilde{y}^2}} C^{-1}g(\widetilde{x}\widetilde{y})\ d\widetilde{x}\ d\widetilde{y} \\
&= \int_{0}^{1/\sqrt{2}} \int_{\widetilde{y}}^{\sqrt{1-\widetilde{y}^2}} f_{X,Y}(\widetilde{x},\widetilde{y})\ d\widetilde{x}\ d\widetilde{y} \\
&= P(Y < X).
\end{aligned}$$

Since the area associated with $\{(x,y)' \in \mathcal{A} : x = y\}$ is zero, we must have $P(X = Y) = 0$. As such, $P(X < Y)$ and $P(Y < X)$ must both equal $1/2$.