STA 623 — Fall 2010 — Dr. Charnigo

In-Class Assessment

This assessment is a strictly individual activity. Textbooks, notes, calculators, computers, and technology with Internet access are prohibited. Record that which you want graded in the blue book.

[20] 1. Students commonly confuse the concepts of "mutually exclusive" and "independence". This exercise provides clarification.

[10] a. For two events A_1 and A_2, state what is meant when we say that they are mutually exclusive and what is meant when we say that they are independent.

[10] b. If $P(A_1) \neq 0$ and $P(A_2) \neq 0$, show that A_1 and A_2 cannot be both independent and mutually exclusive. Hence, "independent" and "mutually exclusive" are not synonyms.

[40] 2. This exercise explores conditions under which a sequence of random variables converges, in some sense, to a constant. In what follows, you may quote without proof the familiar result that the moment generating function of a normal random variable with mean $\mu \in (-\infty, +\infty)$ and standard deviation $\sigma \in (0, +\infty)$ is $\exp[\mu t + t^2\sigma^2/2]$.

[10] a. Define a sequence of random variables $X_1, X_2, \ldots$ such that X_n has the normal distribution with mean 4 and variance $1/n$ for each $n \in \{1, 2, \ldots\}$. To what does the corresponding sequence of moment generating functions converge as $n \to \infty$?

[10] b. Let ϵ be a small but fixed (i.e., not dependent on n) positive number. Use part a to show that $P(|X_n - 4| \leq \epsilon) = P(4 - \epsilon \leq X_n \leq 4 + \epsilon)$ converges to 1 as $n \to \infty$. We say that the sequence of random variables converges "in probability" to the constant 4.

[10] c. Again let ϵ be a small but fixed (i.e., not dependent on n) positive number. However, we no longer assume that X_n has the normal distribution with mean 4 and variance $1/n$ for each $n \in \{1, 2, \ldots\}$. We only assume, at this juncture, that X_n has finite mean and variance for each $n \in \{1, 2, \ldots\}$. State (without proof) Chebychev's Inequality as it applies to $P(|X_n - 4| \geq \epsilon) = P((X_n - 4)^2 \geq \epsilon^2)$.

[10] d. Noting that $E[(X_n - 4)^2] = E[(X_n - E[X_n] + E[X_n] - 4)^2] = Var[X_n] + (E[X_n] - 4)^2$, use part c to conclude that the sequence of random variables converges "in probability" to the constant 4 if $Var[X_n]$ and $E[X_n] - 4$ both converge to 0 as $n \to \infty$. Hence, normality of the X_n was not essential for the conclusion in part b. For example, the conclusion in part b would have held if X_n had been (say) uniformly distributed on $[4 - \sqrt{3/n}, 4 + \sqrt{3/n}]$.

[40] 3. This exercise explores computations of expected values.

[10] a. State (without proof) Jensen's Inequality for a non-degenerate random variable X and a real-valued function g with strictly positive second derivative on an interval containing the support set of X.

[10] b. Let X have the Poisson distribution with mean $\lambda \in (0, \infty)$. What does Jensen's Inequality say about $E[\exp(-X)]$?

[10] c. Let Y have the Poisson distribution with mean $\lambda \exp[-1]$. Adopting the usual notation for probability mass functions, show that $\exp[-x] f_X(x) = c(\lambda) f_Y(x)$ for nonnegative integers x, where $c(\lambda)$ is a function of λ (but not x) that you will specify.

[10] d. Continuing from part c, now apply the kernel method to calculate $E[\exp(-X)]$ and provide a direct confirmation of your answer to part b.