STA 623 – Fall 2010 – Dr. Charnigo

Section 1.1: Set Theory

Sample space. Suppose that we conduct an experiment. The set of all possible outcomes from that experiment is referred to as the sample space. When convenient, we use S as a symbol for the sample space.

Example (sample space). Suppose that I flip a coin four times. Critique the following statement: "The sample space is $\{0, 1, 2, 3, 4\}$, where the number refers to how many times the coin lands on heads."

Event. Suppose that every element of a set A is an element of the sample space S, which we write as $A \subset S$. Then we refer to A as an event.

Example (event). Critique the following statement: "Events are elements of the sample space."

Union, intersection, and complementation operators. Suppose that $A, B \subset S$. We define the union $A \cup B$ as the set containing all elements of S present in A or B or both. We define the intersection $A \cap B$ as the set containing all elements of S present in both A and B. We define the complementation A^c as the set containing all elements of S not present in A.

Example (union, intersection, and complementation operators). Suppose that I roll a six-sided die. The sample space is $\{1,2,3,4,5,6\}$, where the number refers to the result of the roll. Let A be the event that I roll an odd number and B be the event that I roll a multiple of 3. What are $A \cup B$, $A \cap B$, A^c, $A \cup A^c$, $A \cap A^c$, and $(A \cup A^c)^c$?

Properties of operators. Suppose that $A, B, C \subset S$. We have the following.

1. Commutativity: $A \cup B = B \cup A$, $A \cap B = B \cap A$.
2. Associativity: $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$.
3. Distributivity: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
4. DeMorgan laws: $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$.

Example (properties of operators). Critique the following attempt at proof of the first DeMorgan law: "Suppose that $B = A^c$. On the one hand, $A \cup B = S$ and $(A \cup B)^c = S^c = \emptyset$. On the other hand, $B^c = A$ and $A^c \cap B^c = A^c \cap A = \emptyset$. This completes the proof."

Proving that two sets are the same. There is a standard way to prove that two sets are the same: show that each is a subset of the other. For such proofs the following notation is helpful: $\in$ means "is an element of", while $\notin$ means "is not an element of".

Example (proving that two sets are the same). Let us prove the first DeMorgan law. Suppose that $x \in (A \cup B)^c$. Then $x \notin A \cup B$. Hence $x \notin A$ and $x \notin B$. So $x \in A^c$ and $x \in B^c$. Hence $x \in A^c \cap B^c$. This shows that $(A \cup B)^c \subset A^c \cap B^c$. Starting from $x \in A^c \cap B^c$ and working our way back to $x \in (A \cup B)^c$ shows that $A^c \cap B^c \subset (A \cup B)^c$.

General unions and intersections. Suppose that $A_\gamma \subset S$ for all $\gamma \in \Gamma$, where Γ is some non-empty index set. Then we define $\cup_{\gamma \in \Gamma} A_\gamma$ to contain all elements of S present in at least one of the A_γ and $\cap_{\gamma \in \Gamma} A_\gamma$ to contain all elements of S present in all of the A_γ. Common choices for Γ include {1,2}, which reproduces our earlier definitions, and {1,2,3,...}, which gives rise to so-called countable unions and intersections.

Example (general unions and intersections). Suppose that $A_i = [0, i]$ for $i \in \{1, 2, 3, \ldots\}$. What are $\cup_{i=1}^{\infty} A_i$ and $\cap_{i=1}^{\infty} A_i$? For generic $A_1, A_2, A_3, \ldots \subset S$, show that $\cap_{n=1}^{\infty} \cup_{i=n}^{\infty} A_i$ consists of all elements of S that belong to infinitely many of $A_1, A_2, A_3, \ldots$ and that $\cup_{n=1}^{\infty} \cap_{i=n}^{\infty} A_i$ consists of all elements of S that belong to all but finitely many of $A_1, A_2, A_3, \ldots$.

Mutually exclusive and collectively exhaustive. Suppose that $A_\gamma \subset S$ for all $\gamma \in \Gamma$, where Γ is some non-empty index set. If $A_i \cap A_j = \emptyset$ for all unequal $i, j \in \Gamma$, then we say that the A_γ are mutually exclusive. If $\cup_{\gamma \in \Gamma} A_\gamma = S$, then we say that the A_γ are collectively exhaustive. If the A_γ are both mutually exclusive and collectively exhaustive, then we say that they constitute a partition of S.