STA 623 – Fall 2010 – Dr. Charnigo

Section 1.2: Basics of Probability Theory

Axiomatic foundations. Consider the simple experiment of flipping a coin. The sample space S is the finite set $\{Heads, Tails\}$. If we were to repeat this experiment an indefinitely large number of times, then we might find that about half of the flips resulted in Heads while about half of the flips resulted in Tails. This might motivate us to define the probability of getting a Heads on a single flip as 1/2 and likewise to define the probability of getting a Tails on a single flip as 1/2. In other words, we might define the probability of an event in an experiment based on the event's relative frequency over an indefinitely large number of repetitions of that experiment.

The relative frequency perspective helps us to describe to the lay person what probability is. However, most statisticians prefer to define probability axiomatically. Therefore, we proceed axiomatically for now. A connection to the relative frequency perspective will be made explicit in STA 606, when you encounter the Laws of Large Numbers.

Let $P(A)$ denote the probability of an event $A \subset S$. We want the following conditions or "axioms" to be satisfied for any events $A, A_1, A_2, \ldots \subset S$.

1. $P(A) \geq 0$.
2. $P(S) = 1$.
3. $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$ whenever $A_i \cap A_j = \emptyset$ for $i \neq j$.

Unfortunately, there are complications when S is uncountably infinite. These complications are circumvented by requiring that probabilities be defined and axioms hold not for all events but only for events belonging to a special collection of subsets of S called a "sigma algebra" or a "sigma field".

Formally, a sigma field $\mathcal{B}$ is a collection of subsets of S satisfying the following three properties.

a. $\emptyset \in \mathcal{B}$.

b. If $A \in \mathcal{B}$, then $A^c \in \mathcal{B}$.

c. If $A_1, A_2, \ldots \in \mathcal{B}$, then $\cup_{i=1}^{\infty} A_i \in \mathcal{B}$.

Example (Axiomatic foundations). Suppose that $S = \{1, 2, 3, 4, 5, 6\}$, as if I were rolling a six-sided die. What is the smallest sigma field containing $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, and $\{6\}$? How can we define probabilities for the events in this sigma field so that the axioms are satisfied? What is the smallest sigma field containing $\{3, 6\}$ and $\{2, 4, 6\}$?

Calculus of probabilities. Several useful results follow from the axioms. We assume that $A, B, C_1, C_2, \ldots \in \mathcal{B}$.

4. $P(\emptyset) = 0$.
5. $P(A) \leq 1$.
6. $P(A^c) = 1 - P(A)$.
7. $P(B \cap A^c) = P(B) - P(B \cap A)$.
8. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
9. If $A \subset B$, then $P(A) \leq P(B)$.
10. $P(\cup_{i=1}^{\infty} C_i) \leq \Sigma_{i=1}^{\infty} P(C_i)$.
11. If $C_1, C_2, \ldots$ is a partition, then $P(A) = \Sigma_{i=1}^{\infty} P(A \cap C_i)$.

Example (Calculus of probabilities). Let us prove result 4. Put $A_i := \emptyset$ for $i \in \{1, 2, \ldots\}$. Then $A_i \cap A_j = \emptyset$ for $i \neq j$ and $\cup_{i=1}^{\infty} A_i = \emptyset$. By axiom 3 we have $P(\emptyset) = \Sigma_{i=1}^{\infty} P(\emptyset)$, which is only possible if $P(\emptyset) = 0$.

Critique the following attempt at proof of result 10: "The left hand side must be less than or equal to 1 by result 5, while the right hand side can be infinite. So the left hand side is less than or equal to the right hand side."

Let us prove result 10. Put $D_1 := C_1$, $D_2 := C_2 \cap C_1^c$, $D_3 := C_3 \cap C_1^c \cap C_2^c$, $D_4 := C_4 \cap C_1^c \cap C_2^c \cap C_3^c$, and so forth. (Are $D_1, D_2, \ldots \in \mathcal{B}$?)

Then $\cup_{i=1}^{\infty} D_i = \cup_{i=1}^{\infty} C_i$. (Suppose that $x \in \cup_{i=1}^{\infty} C_i$. Let j be the smallest positive integer for which $x \in C_j$. Then $x \in D_j$, so that $x \in \cup_{i=1}^{\infty} D_i$. This shows that $\cup_{i=1}^{\infty} C_i \subset \cup_{i=1}^{\infty} D_i$. How do we know that $\cup_{i=1}^{\infty} D_i \subset \cup_{i=1}^{\infty} C_i$?)

Moreover, $D_i \cap D_j = \emptyset$ for $i \neq j$. (Suppose that $x \in D_i \cap D_j$. Then $x \in C_i$ and $x \in C_j$. If $i > j$, then $x \in C_j$ implies $x \notin C_j^c$ and $x \notin C_i \cap C_1^c \cap \cdots \cap C_{i-1}^c = D_i$. This is a contradiction, so there is no such x. Likewise, we obtain a contradiction if $i < j$.)

By axiom 3 we have $P(\cup_{i=1}^{\infty} D_i) = \Sigma_{i=1}^{\infty} P(D_i)$. By result 9 we have $P(D_i) \leq P(C_i)$ for $i \in \{1, 2, \ldots\}$. Since $\cup_{i=1}^{\infty} D_i = \cup_{i=1}^{\infty} C_i$, we conclude that $P(\cup_{i=1}^{\infty} C_i) = P(\cup_{i=1}^{\infty} D_i) = \Sigma_{i=1}^{\infty} P(D_i) \leq \Sigma_{i=1}^{\infty} P(C_i)$.

Counting techniques. How many ways are there to select r objects from a collection of n objects, where r and n ($> r$) are positive integers? The answer depends on whether objects can be replaced and on whether the order in which objects are selected matters.

First consider the situation in which objects can be replaced and the order in which objects are selected matters. An example of this is choosing a four-digit PIN number. Here we have $r = 4$ and $n = 10$. The same digit can appear more than once (3934 is a valid four-digit PIN number), and the order of the digits matters (3934 is not the same as 9334). In this example we see readily that there are $10000 = 10^4$ possibilities for a four-digit PIN number. The general rule is that there are n^r possibilities when objects can be replaced and the order in which objects are selected matters.

Now consider the situation in which objects cannot be replaced and the order in which objects are selected matters. An example of this is being asked to choose your favorite and second favorite colors from among red, orange, yellow, green, blue, and purple. Here we have $r = 2$ and $n = 6$. The same color cannot appear more than once (red cannot be both your favorite and your second favorite color), and the order of the colors matters (red as a favorite and yellow as a second favorite is not the same as yellow as a favorite and red as a second favorite). In this example we see readily that there are 30 possibilities. The general rule is that there are $n \times (n-1) \times \cdots \times (n-r+1) = n!/(n-r)!$ possibilities when objects cannot be replaced and the order in which objects are selected matters. Here we use $n!$ (read "n factorial") to represent $n \times (n-1) \times (n-2) \times \cdots \times 1$ for any positive integer n. By convention $0! = 1$.

Next consider the situation in which objects cannot be replaced and the order in which objects are selected does not matter. An example of this is a lottery ticket in which you win by matching the numbers on 6 balls drawn without replacement from a vat containing 44 balls. Here we have $r = 6$ and $n = 44$. The same number cannot appear twice (once ball 1 is removed from the vat, ball 1 cannot be drawn from the vat again), and the order of the numbers does not matter (if your lottery ticket shows $7, 11, 12, 18, 35, 42$, you still win if the balls are drawn in the order $42, 35, 18, 12, 11, 7$). In this example there are $(44 \times 43 \times 42 \times 41 \times 40 \times 39)/(6 \times 5 \times 4 \times 3 \times 2 \times 1) = 44!/\{6!\,38!\}$ possibilities. The division by $6!$ avoids overcounting (without it we would have $6!$ copies of $7, 11, 12, 18, 35, 42$ since $7, 11, 12, 18, 35, 42$ is the same as $42, 35, 18, 12, 11, 7$, which is the same as $7, 12, 35, 11, 18, 42$, and so forth). The general rule is that there are $n!/\{r!(n-r)!\} =: \binom{n}{r}$ (read "n choose r") possibilities when objects cannot be replaced and the order in which objects are selected does not matter.

Finally consider the situation in which objects can be replaced and the order in which objects are selected does not matter. An example of this is if we list from lowest to highest your scores on the five written assignments this semester. Here we have $r = 5$ and $n = 101$ because the possible scores are $0, 1, 2, \ldots, 100$. You can get the same score more than once (all of you are hoping to get more than one 100), but the order in which you acquire the scores does not matter (100, 95, 90, 85, 80 and 90, 85, 80, 95, 100 are both listed as 80, 85, 90, 95, 100). Although not obvious, in this example there are $105!/\{100!\,5!\}$ possibilities. The general rule is that there are $\binom{n+r-1}{r}$

possibilities when objects can be replaced and the order in which objects are selected does not matter.

Enumerating outcomes. Counting techniques are useful in problems where S is finite and all elements of S are equally likely. Indeed, suppose that $S = \{s_1, s_2, \ldots, s_n\}$ for some positive integer n, that $\mathcal{B}$ consists of all subsets of S, and that $P(\{s_i\}) = 1/n$ for $i \in \{1, 2, \ldots, n\}$. We apply probability axiom 3 to find that $P(A) = \sum_{s_i \in A} P(\{s_i\}) = \sum_{s_i \in A} 1/n = card(A)/card(S)$, where *card* denotes the cardinality of (i.e., number of elements in) a set.

Example (enumerating outcomes). Consider being dealt a five-card poker hand from a standard deck of 52 playing cards (there are 13 denominations — ace, king, queen, etc. — in each of 4 suits — hearts, diamonds, spades, clubs). Assuming that the deck is well shuffled, what is the probability of being dealt a "full house" (a hand with a pair — two cards of matching denomination — and a triple — three cards of matching denomination)?

To answer this question, we note that the order in which the cards are dealt does not matter and that they are drawn from the deck without replacement. Letting S denote the set of all possible five-card poker hands, we find that $card(S) = \binom{52}{5} = 2598960$.

Letting A denote the set of all five-card poker hands containing a pair and a triple, we can find $card(A)$ by noting that there are ways to specify the denomination for the pair, ways to specify the denomination for the triple, ways to specify suits for the pair, and ways to specify suits for the triple. Multiplying these numbers together gives us $card(A) = 3744$, so that the probability of being dealt a full house is $card(A)/card(S) = 3744/2598960 \approx 0.144\%$.