STA 623 – Fall 2010 – Dr. Charnigo

Section 2.4: Differentiating under an integral sign

The main result. Suppose there exist $\delta > 0$ and $A \in \mathcal{B}^1$ such that the following conditions are met for $t \in (y - \delta, y + \delta)$.

1. The integral $\int_A g(x,t)\, dx =: u(t)$ is absolutely convergent.
2. For any fixed $x \in A$, $\frac{\partial}{\partial t} g(x,t)$ exists and is a continuous function of t.
3. There exists $h(x) \geq 0$ such that $|\frac{\partial}{\partial t} g(x,t)| \leq h(x)$ and $\int_A h(x)\, dx < \infty$.

Then $\frac{d}{dt} u(t)|_{t=y}$ exists and equals $\int_A \frac{\partial}{\partial t} g(x,t)|_{t=y}\, dx$.

Application to moment generating function. Let X be a continuous random variable with probability density function $f_X(x)$ supported on $\mathcal{X} \subset \mathbb{R}$. Assume there exists $\epsilon > 0$ such that

$$M_X(t) := E[\exp(tX)] = \int_{\mathcal{X}} \exp(tx)\ f_X(x)\ dx < \infty$$

for all $t \in [-\epsilon, \epsilon]$. We wish to show that

$$\frac{d}{dt} M_X(t)|_{t=0} = \int_{\mathcal{X}} \frac{\partial}{\partial t} \exp(tx)|_{t=0}\ f_X(x)\ dx = E[X].$$

Put $\delta := \epsilon/2$, $A := \mathcal{X}$, $y := 0$, and $g(x,t) := \exp(tx) f_X(x)$.

1. Since $(-\epsilon/2, \epsilon/2) \subset [-\epsilon, \epsilon]$ we have

$$\int_A g(x,t)\ dx = \int_{\mathcal{X}} \exp(tx)\ f_X(x)\ dx < \infty$$

for $t \in (y - \delta, y + \delta) = (-\epsilon/2, \epsilon/2)$, by assumption.

2. For any fixed $x \in A = \mathcal{X}$ we have

$$\frac{\partial}{\partial t} g(x,t) = \frac{\partial}{\partial t} \exp(tx) f_X(x) = x \exp(tx) f_X(x),$$

which is obviously continuous in t.

3. For $t \in (y - \delta, y + \delta) = (-\epsilon/2, \epsilon/2)$, we have

$$|x \exp(tx)| \leq x \exp(\epsilon x/2) 1_{\{x \geq 0\}} - x \exp(-\epsilon x/2) 1_{\{x < 0\}}.$$

There exists $C > 0$ such that, when $|x| \geq C$, we have

$$\begin{aligned} & x\exp(\epsilon x/2)1_{\{x\geq 0\}} - x\exp(-\epsilon x/2)1_{\{x<0\}} \\ \leq \ & \exp(\epsilon x)1_{\{x\geq 0\}} + \exp(-\epsilon x)1_{\{x<0\}} \\ \leq \ & \exp(\epsilon x) + \exp(-\epsilon x). \end{aligned}$$

On the other hand, when $|x| < C$, we have

$$\begin{aligned} & x\exp(\epsilon x/2)1_{\{x\geq 0\}} - x\exp(-\epsilon x/2)1_{\{x<0\}} \\ \leq \ & C\exp(\epsilon C/2)1_{\{x\geq 0\}} + C\exp(\epsilon C/2)1_{\{x<0\}} \\ = \ & C\exp(\epsilon C/2). \end{aligned}$$

Hence, for all $x \in \mathcal{X}$, we have

$$|x\exp(tx)| \leq \exp(\epsilon x) + \exp(-\epsilon x) + C\exp(\epsilon C/2).$$

How do we finish?

A companion result for summation. Suppose there exists $\delta > 0$ such that the following conditions are met for $t \in (y-\delta, y+\delta)$.

1. The summation $\sum_{x=0}^{\infty} g(x,t) =: u(t)$ is absolutely convergent.
2. For any fixed $x \in \{0,1,2,\ldots\}$, $\frac{\partial}{\partial t} g(x,t)$ exists and is a continuous function of t.
3. There exists $h(x) \geq 0$ such that $|\frac{\partial}{\partial t} g(x,t)| \leq h(x)$ and $\sum_{x=0}^{\infty} h(x) < \infty$.

Then $\frac{d}{dt}u(t)|_{t=y}$ exists and equals $\sum_{x=0}^{\infty} \frac{\partial}{\partial t} g(x,t)|_{t=y}$.

Application to geometric series. We know that

$$\sum_{x=0}^{\infty} q^x = \frac{1}{1-q}$$

for $q \in (0,1)$. We would like to differentiate in q to conclude that

$$\sum_{x=0}^{\infty} x q^{x-1} = \frac{1}{(1-q)^2} \tag{1}$$

and

$$\sum_{x=0}^{\infty} x(x-1) q^{x-2} = \frac{2}{(1-q)^3}, \tag{2}$$

as these would be useful results for evaluating $E[X]$ and $E[X(X-1)]$ if X had the probability mass function $f_X(x) := (1-q)q^x$ for $x \in \{0, 1, 2, \ldots\}$. We will verify (1) together. Verification of (2) is left to you. Let y be a fixed element of $(0,1)$, let $\delta := \min\{y/2, (1-y)/2\}$, and put $g(x,q) := q^x$.

1. Since $(y-\delta, y+\delta) \subset (0,1)$, we have

$$\sum_{x=0}^{\infty} g(x,q) = \sum_{x=0}^{\infty} q^x < \infty$$

for $q \in (y-\delta, y+\delta)$.

2. For any fixed $x \in \{0, 1, 2, \ldots\}$, we have

$$\frac{\partial}{\partial q} g(x,q) = \frac{\partial}{\partial q} q^x = x q^{x-1} = x \exp[(x-1)\log q],$$

which is obviously continuous in $q \in (y-\delta, y+\delta)$.

3. For $q \in (y-\delta, y+\delta)$ and $x \in \{0, 1, 2, \ldots\}$, we have

$$x \exp[(x-1)\log q] \le x \exp[(x-1)\log(y+\delta)].$$

There exists a positive integer C such that $x \exp[(x-1)\log(y+\delta)]$ is strictly decreasing as a function of $x \in [C, \infty)$. Hence,

$$\sum_{x=C+1}^{\infty} x \exp[(x-1)\log(y+\delta)] \le \int_{C}^{\infty} x \exp[(x-1)\log(y+\delta)]\, dx,$$

and the latter is obviously finite. How do we finish?