STA 623 – Fall 2010 – Dr. Charnigo

Section 3.5: Location and Scale Families

Location family. Let $f(x)$ be any probability density function, and let μ be any real number. Then $f(x;\mu) := f(x-\mu)$ is also a probability density function because, upon substitution of $y := x-\mu$ and $dy := dx$,

$$\int_{-\infty}^{\infty} f(x-\mu)\ dx = \int_{-\infty}^{\infty} f(y)\ dy = 1.$$

We refer to $\{f(x;\mu) : \mu \in \mathbb{R}\}$ as a location family.

Examples (location family). [Normal location family] Put

$$f(x) := (2\pi)^{-1/2} \exp[-x^2/2],$$

and let μ be any real number. Then

$$f(x;\mu) := f(x-\mu) = (2\pi)^{-1/2} \exp[-(x-\mu)^2/2].$$

Geometrically, $f(x;\mu)$ is a rigid horizontal shift of $f(x)$ by the amount μ. In fact, if $X \sim f(x)$, then $Y := X + \mu \sim f(y;\mu)$ because with $g(x) := x + \mu$ we have $g^{-1}(y) = y - \mu$ and $\frac{d}{dy} g^{-1}(y) = 1$. We also have $E[Y] = E[X] + \mu = \mu$, while $Var[Y] = Var[X] = 1$.

[Exponential location family] Put

$$f(x) := \exp[-x] 1_{x \geq 0},$$

and let μ be any real number. Then

$$f(x;\mu) := f(x-\mu) = \exp[-(x-\mu)] 1_{x \geq \mu}.$$

Again, if $X \sim f(x)$, then $Y := X + \mu \sim f(y;\mu)$. Two questions:

1. Suppose you were told that $Y \sim f(y;\mu)$ but not given the value of μ. If you observed $Y = 2$, what would be your guess (or "estimate") of μ?

2. Is the location family $\{f(x;\mu) : \mu \in \mathbb{R}\}$ an exponential family?

[Cauchy location family] Put

$$f(x) := \pi^{-1}/(1+x^2),$$

and let μ be any real number. Then

$$f(x;\mu) := f(x-\mu) = \pi^{-1}/(1+(x-\mu)^2).$$

Once more, if $X \sim f(x)$, then $Y := X + \mu \sim f(y;\mu)$. Two questions:

1. Do we have $E[Y] = E[X] + \mu$?

2. Define the $100p^{th}$ percentile for the distribution of X to be the number x_p such that $P(X \leq x_p) = p$. What are the 25^{th}, 50^{th}, and 75^{th} percentiles for the distribution of X? If we analogously define percentiles for the distribution of Y, how are they related to those for the distribution of X?

Scale family. Let $f(x)$ be any probability density function, and let σ be any positive real number. Then, as you should verify, $f(x;\sigma) := f(x/\sigma)/\sigma$ is also a probability density function. We refer to $\{f(x;\sigma) : \sigma \in (0,\infty)\}$ as a scale family.

Examples (scale family). [Normal scale family] Put

$$f(x) := (2\pi)^{-1/2}\exp[-x^2/2],$$

and let σ be any positive real number. Then

$$f(x;\sigma) := f(x/\sigma)/\sigma = (2\pi)^{-1/2}\sigma^{-1}\exp[-x^2/(2\sigma^2)].$$

Geometrically, $f(x;\sigma)$ is a compression of $f(x)$ if $\sigma < 1$ and a dilation of $f(x)$ if $\sigma > 1$. If $X \sim f(x)$, then, as you should verify, $Y := \sigma X \sim f(y;\sigma)$. We have $E[Y] = E[\sigma X] = \sigma E[X] = 0$ and $Var[Y] = Var[\sigma X] = \sigma^2 Var[X] = \sigma^2$.

[Exponential scale family] Put

$$f(x) := \exp[-x]1_{x\geq 0},$$

and let σ be any positive real number. Then

$$f(x;\sigma) := f(x/\sigma)/\sigma = \sigma^{-1}\exp[-x/\sigma]1_{x\geq 0}.$$

Again, if $X \sim f(x)$, then $Y := \sigma X \sim f(y;\sigma)$.

[Cauchy scale family] Put

$$f(x) := \pi^{-1}/(1+x^2),$$

and let σ be any positive real number. Then

$$f(x;\sigma) := f(x/\sigma)/\sigma = (\pi\sigma)^{-1}/(1+(x/\sigma)^2).$$

Once more, if $X \sim f(x)$, then $Y := \sigma X \sim f(y;\sigma)$. How do the 25^{th}, 50^{th}, and 75^{th} percentiles for the distribution of X relate to those for the distribution of Y?

Location-scale family. Let $f(x)$ be any probability density function. Let μ be any real number, and let σ be any positive real number. Then, as you should verify, $f(x;\mu,\sigma) := f((x-\mu)/\sigma)/\sigma$ is also a probability density function. We refer to $\{f(x;\mu,\sigma) : \mu \in \mathbb{R}, \sigma \in (0,\infty)\}$ as a location-scale family. See Written Assignment 4 from Fall 2009 for an example.

Further remarks. Not all two-parameter families of distributions are location-scale families. (Consider the family of gamma distributions.)

The concept of location-scale families is not particularly useful for discrete random variables because, for the most part, the support sets for different distributions within the same location-scale family would have no elements in common. Consequently, the location-scale family would not coincide with any recognizable parametric family. (Consider building a location-scale family starting from a Poisson probability mass function. If $\mu = 2.6$ and $\sigma = 1.5$, then the support set is $\{2.6, 4.1, 5.6, 7.1, \ldots\}$, which does not contain any integers.)