STA 623 – Fall 2010 – Dr. Charnigo

Section 4.3: Bivariate Transformations

One-to-one bivariate transformation formula. Let $\mathbf{X}$ be a continuous random vector with components X and Y, and let $f_{X,Y}(x,y)$ denote the joint probability density function of X and Y. Suppose that $g_1(x,y)$, $g_2(x,y)$ are (appropriately measurable) functions that define a one-to-one bivariate transformation, in the sense that $g_1(x,y) = g_1(w,z), g_2(x,y) = g_2(w,z)$ implies $x = w, y = z$. Then the equations $u = g_1(x,y), v = g_2(x,y)$ can be solved for x and y, say $x = h_1(u,v)$ and $y = h_2(u,v)$. Put $U := g_1(X,Y)$ and $V := g_2(X,Y)$. Assuming the existence of all partial derivatives referenced below, the joint probability density function of U and V is

$$f_{U,V}(u,v) = f_{X,Y}(h_1(u,v), h_2(u,v)) \left| Det \begin{bmatrix} \frac{\partial h_1(u,v)}{\partial u} & \frac{\partial h_1(u,v)}{\partial v} \\ \\ \frac{\partial h_2(u,v)}{\partial u} & \frac{\partial h_2(u,v)}{\partial v} \end{bmatrix} \right| 1_{\{(u,v)' \in S_{U,V}\}},$$

where $Det[\cdot]$ returns the determinant of a matrix (assumed nonzero here) and

$$S_{U,V} := \{(u,v)' \in \mathbb{R}^2 : \exists (x,y)' \in \mathbb{R}^2 \text{ with } u = g_1(x,y), v = g_2(x,y), f_{X,Y}(x,y) > 0\}.$$

Since any number may be regarded as a 1×1 matrix, the one-to-one bivariate transformation formula above is readily seen to be an extension of the one-to-one univariate transformation formula from Section 2.1,

$$f_U(u) = f_X(h(u)) \left| Det \left[\frac{dh(u)}{du} \right] \right| 1_{\{u \in S_U\}},$$

where $U := g(X)$, $h(u) := g^{-1}(u)$, $f_X(x)$ is the probability density function for X, $f_U(u)$ is the probability density function for U, and

$$S_U := \{u \in \mathbb{R} : \exists x \in \mathbb{R} \text{ with } u = g(x), f_X(x) > 0\}.$$

Example (one-to-one bivariate transformation formula). Suppose that X has the chi-square distribution on 2 df,

$$f_X(x) = (1/2)\exp[-x/2]1_{\{x>0\}},$$

and that, independently, Y has the uniform distribution on $(-\pi/2, \pi/2)$,

$$f_Y(y) = (1/\pi)1_{\{-\pi/2<y<\pi/2\}}.$$

Put $g_1(x, y) := \sqrt{x}\cos y$ and $g_2(x, y) := \sqrt{x}\sin y$ for $x \in (0, \infty)$ and $y \in (-\pi/2, \pi/2)$. Let $U := g_1(X, Y)$ and $V := g_2(X, Y)$. What is the joint distribution of U and V? What are the marginal distributions of U and V?

Step 1. Find the support of U and V. Since $\cos y$ must be positive when $-\pi/2 < y < \pi/2$ while $\sin y$ can be positive or negative or zero, we have $S_{U,V} = \{(u, v)' \in \mathbb{R}^2 : u > 0\}$.

Step 2. Verify that the transformation is one-to-one. With $u = \sqrt{x}\cos y$ and $v = \sqrt{x}\sin y$, we have $x = u^2+v^2$ and $\tan[y] = v/u$. Since $-\pi/2 < y < \pi/2$, $\tan[y] = v/u$ has the unique solution $y = \arctan[v/u]$. So put $h_1(u, v) := u^2+v^2$ and $h_2(u, v) := \arctan[v/u]$. The fact that we were able to solve for y and x not only implies that the transformation is one-to-one but also provides useful results for the next step.

Step 3. Evaluate the matrix determinant. We have

$$\frac{\partial h_1(u, v)}{\partial u} = 2u, \quad \frac{\partial h_1(u, v)}{\partial v} = 2v,$$

$$\frac{\partial h_2(u, v)}{\partial u} = \frac{1}{1 + (v/u)^2}\frac{\partial(v/u)}{\partial u} = \frac{-v}{u^2 + v^2}, \quad \text{and}$$

$$\frac{\partial h_2(u, v)}{\partial v} = \frac{1}{1 + (v/u)^2}\frac{\partial(v/u)}{\partial v} = \frac{u}{u^2 + v^2}.$$

So the matrix determinant is

Step 4. Report the joint probability density function. We have

$$f_{X,Y}(h_1(u, v), h_2(u, v)) =$$

so that

$f_{U,V}(u, v) =$

Step 5. Report the marginal probability density functions. Since $f_{U,V}(u, v)$ can be written in the form $g(u)h(v)$, the kernels of $f_U(u)$ and $f_V(v)$ are obvious. All we need to do is determine the normalizing constants, but this is not difficult. Since V is obviously a standard normal random variable, we must have

$$f_V(v) = (2\pi)^{-1/2} \exp[-v^2/2].$$

This implies that

$$f_U(u) = 2(2\pi)^{-1/2} \exp[-u^2/2] 1_{\{u>0\}}.$$

How would you describe the distribution of U?

Remarks. If the above example were changed by taking Y to have the uniform distribution on $(\pi/2, 3\pi/2)$, then the computations would remain almost the same and we would end up with

$$f_V(v) = (2\pi)^{-1/2} \exp[-v^2/2],$$

$$f_U(u) = 2(2\pi)^{-1/2} \exp[-u^2/2] 1_{\{u<0\}}.$$

If Y had the uniform distribution on $(-\pi/2, 3\pi/2)$, what do you think the marginal distributions of V and U should be? The computations would be more difficult, however, since $\tan y = v/u$ would not uniquely determine y. How would you overcome that difficulty?