STA 623 — Fall 2010 — Dr. Charnigo

Solutions to Written Assignment 1

1. Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. Since $x \in B \cup C$, either $x \in B$ or $x \in C$. In the first case, $x \in A \cap B$. In the second case, $x \in A \cap C$. Hence $x \in (A \cap B) \cup (A \cap C)$. This shows that $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$.

Let $x \in (A \cap B) \cup (A \cap C)$. Then either $x \in A \cap B$ or $x \in A \cap C$. In the first case, $x \in A$ and $x \in B$. In the second case, $x \in A$ and $x \in C$. Hence $x \in A$ and $x \in B \cup C$. So $x \in A \cap (B \cup C)$. This shows that $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$.

2. Put $A_1 := A$, $A_2 := A^c$, and $A_j := \emptyset$ for $j \in \{3, 4, \ldots\}$. Since $A \cup A^c = S$ and $A \cap A^c = \emptyset$, we have that $\cup_{j=1}^{\infty} A_j = S$ and $A_i \cap A_j = \emptyset$ for $i \neq j$. As such, axiom 3 yields

$$P(S) = P(\cup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} P(A_j).$$

By axiom 2 $P(S) = 1$ and by result 4 $P(A_j) = P(\emptyset) = 0$ for $j \in \{3, 4, \ldots\}$. We are left with

$$1 = P(A_1) + P(A_2) = P(A) + P(A^c).$$

Since $P(A^c) \geq 0$ by axiom 1, we conclude that $P(A) \leq 1$.

Remark: Note that this argument proves results 5 and 6 simultaneously.

3. We have

$$P(A \cup B \cup C) = P(A \cup D) = P(A) + P(D) - P(A \cap D)$$

from result 8 for calculating probabilities. Exercise 1 showed that

$$A \cap D = A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$$

so that invoking result 8 once again yields

$$P(A\cap D) = P((A\cap B)\cup(A\cap C)) = P(A\cap B)+P(A\cap C)-P((A\cap B)\cap(A\cap C)) = P(A\cap B)+P(A\cap C)-P(A\cap B\cap C).$$

A final invocation of result 8 yields

$$P(D) = P(B \cup C) = P(B) + P(C) - P(B \cap C).$$

Putting all of these pieces together, we find that

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

4. A good Venn diagram would depict $A \cup B \cup C$ as the union of seven mutually exclusive sets ($A \cap B \cap C$, $A \cap B \cap C^c$, $A \cap B^c \cap C$, $A \cap B^c \cap C^c$, $A^c \cap B \cap C$, $A^c \cap B \cap C^c$, $A^c \cap B^c \cap C$) and indicate that the formula from exercise 3 provides a net count of one for each of these sets. For example, the formula from exercise 3 counts $A \cap B \cap C$ in each of its seven terms, but four of the terms have positive signs and three have negative signs, while $A \cap B \cap C^c$ is counted in the first, second, and fourth terms, of which two have positive signs and one has a negative sign.

5. There are $\binom{13}{2} = 78$ ways to select the denominations for the two triples (not 13×12 because, for instance, a triple of 9's and a triple of 10's is the same as a triple of 10's and a triple of 9's), then 11 ways to select the denomination for the pair, and then $\binom{10}{2} = 45$ ways to select the denominations for

the two singles. For each triple there are $\binom{4}{3} = 4$ ways to select the suits, for the pair there are $\binom{4}{2} = 6$ ways to select the suits, and for each single there are $\binom{4}{1} = 4$ ways to select the suits. So there are $78 \times 11 \times 45 \times 4^2 \times 6 \times 4^2 = 59,304,960$ possible 10-card hands with two triples, one pair, and two singles. On the other hand, there are $\binom{52}{10} = 15,820,024,220$ possible 10-card hands without regard to singles, pairs, etc. With a well-shuffled deck, we may assume that all of these 10-card hands are equally likely, so that the probability of receiving a 10-card hand with two triples, one pair, and two singles is $59,304,960/15,820,024,220 \approx 0.0037$.

6. Since the order in which the balls are drawn from the vat does not matter, there are $\binom{40}{5} = 658,008$ possible draws from the vat. There is only one way for me to win a large prize, but there are $\binom{5}{4} \times \binom{35}{1} = 175$ ways for me to win a small prize since there are $\binom{5}{4}$ ways to pick four numbers from among those on my ticket and $\binom{35}{1}$ ways to pick one number from among those not on my ticket. Assuming that all 658,008 possible draws from the vat are equally likely, the probability that I win a prize is $176/658,008 = 0.00027$. Hence, the probability that I have spent a dollar with nothing to show for it is $1 - 0.00027 = 0.99973$.

7. Let $\mathcal{F} := \{\{1,5\},\{2,4\},\{3\},\{6\}\}$ and $\mathcal{G} := \{\{2,4,6\},\{3,6\}\}$. Also, let $\sigma(\mathcal{F})$ and $\sigma(\mathcal{G})$ denote the smallest sigma fields containing $\mathcal{F}$ and $\mathcal{G}$ respectively.

Recall that sigma fields are closed under complementation, finite or countable unions, and (hence, by DeMorgan's law) finite or countable intersections. As such, any sigma field containing $\mathcal{G}$ must contain $\{6\} = \{2,4,6\} \cap \{3,6\}$, $\{1,2,3,4,5\} = \{6\}^c$, $\{3\} = \{1,2,3,4,5\} \cap \{3,6\}$, $\{2,4\} = \{1,2,3,4,5\} \cap \{2,4,6\}$, and $\{1,5\} = (\{2,4,6\} \cup \{3,6\})^c$. In particular, $\mathcal{F} \subset \sigma(\mathcal{G})$. As such, $\sigma(\mathcal{F}) \subset \sigma(\mathcal{G})$. (Proof: If not, then $\sigma(\mathcal{F}) \cap \sigma(\mathcal{G})$ is a smaller sigma field containing $\mathcal{F}$ than $\sigma(\mathcal{F})$, which contradicts the definition of $\sigma(\mathcal{F})$.)

Likewise, any sigma field containing $\mathcal{F}$ must contain $\{2,4,6\} = \{2,4\} \cup \{6\}$ and $\{3,6\} = \{3\} \cup \{6\}$. In particular, $\mathcal{G} \subset \sigma(\mathcal{F})$. As such, $\sigma(\mathcal{G}) \subset \sigma(\mathcal{F})$.

Having shown that $\sigma(\mathcal{G}) = \sigma(\mathcal{F})$, we now show that $\sigma(\mathcal{F}) = \mathcal{H}$, defined as the collection of all possible unions of $\{1,5\}$,$\{2,4\}$,$\{3\}$,$\{6\}$, and $\emptyset$.

Clearly, $\mathcal{H} \subset \sigma(\mathcal{F})$ because sigma fields are closed under countable unions.

On the other hand, $\mathcal{H}$ is itself a sigma field. First, $\mathcal{H}$ includes $\emptyset$. Second, $\mathcal{H}$ is closed under countable unions. Indeed, let $A_1, A_2, \ldots \in \mathcal{H}$. Then A_1 may be expressed in the form $B_1 \cup B_2 \cup B_3 \cup B_4$, where $B_1, B_2, B_3, B_4 \in \mathcal{F} \cup \{\emptyset\}$, A_2 may be expressed in the form $B_5 \cup B_6 \cup B_7 \cup B_8$, and so forth. Hence, $\cup_{j=1}^{\infty} A_j = \cup_{j=1}^{\infty} B_j \in \mathcal{H}$. Third, $\mathcal{H}$ is closed under complementation. Indeed, let $A \in \mathcal{H}$ and let $C_1 := \{1,5\} \in \mathcal{F}$, $C_2 := \{2,4\} \in \mathcal{F}$, $C_3 := \{3\} \in \mathcal{F}$, $C_4 := \{6\} \in \mathcal{F}$. We can express A in the form $\cup_{j \in I} C_j$, where I is a subset of $\{1,2,3,4\}$. (If I is empty, then the union is defined to be $\emptyset$.) Then A^c has the form $\cup_{j \in I^c} C_j$, where I^c is the complement of I with respect to $\{1,2,3,4\}$. Hence, $A^c = \cup_{j \in I^c} C_j \in \mathcal{H}$.

Since $\mathcal{H}$ is a sigma field containing $\mathcal{F}$, $\mathcal{H}$ must also contain $\sigma(\mathcal{F})$. Therefore, $\mathcal{H} = \sigma(\mathcal{F}) = \sigma(\mathcal{G})$.

8. Let A denote the event that a person has HIV and B the event that a person tests positive for HIV. Applying Bayes' Theorem, we have

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + [1 - P(B^c|A^c)][1 - P(A)]}.$$

Substituting 0.007 for $P(A)$, 0.90 for $P(B|A)$, and $P/100$ for $P(B^c|A^c)$, we obtain

$$P(A|B) = \frac{0.0063}{0.0063 + [1 - P/100]0.993}.$$

If $P(A|B)$ is set to 0.5, then we can solve for P. We have $0.0063 + [1 - P/100]0.993 = 0.0126$, then $[1 - P/100].993 = 0.0063$, then $[1 - P/100] = 0.0063/0.993$, then $[P/100] = 1 - 0.0063/0.993$, and finally $P = 100 - 0.63/0.993 = 99.4$. The test must have almost perfect specificity for a positive predictive value as modest as 0.5!