STA 623 — Fall 2010 — Dr. Charnigo

Solutions to Written Assignment 2

1a. Put $B := \{1\}$, which belongs to the sigma field on $\mathbb{R}$. Then $X^{-1}(B) = \{1\}$, which does not belong to the sigma field on S. Hence, X is not a random variable.

1b. Suppose that B belongs to the sigma field on $\mathbb{R}$. If neither -1 nor $+1$ belongs to B, then $Y^{-1}(B) = \emptyset$. If -1 but not $+1$ belongs to B, then $Y^{-1}(B) = \{1,3,5\} = \{1,5\} \cup \{3\}$. If $+1$ but not -1 belongs to B, then $Y^{-1}(B) = \{2,4,6\} = \{2,4\} \cup \{6\}$. If both -1 and $+1$ belong to B, then $Y^{-1}(B) = S$. In all four cases, $Y^{-1}(B)$ belongs to the sigma field on S. Hence, Y is a random variable.

1c. We have $P(Y = 1) = P(\{\omega \in S : Y(\omega) = 1\}) = P(\{2,4,6\}) = P(\{2,4\}) + P(\{6\}) = 1/6 + 1/6 = 1/3$ and $P(Y = -1) = P(\{\omega \in S : Y(\omega) = -1\}) = P(\{1,3,5\}) = P(\{1,5\}) + P(\{3\}) = 1/3 + 1/3 = 2/3$.

1d. The cumulative distribution function is $(2/3)1_{\{y \geq -1\}} + (1/3)1_{\{y \geq 1\}}$.

2. The limit of the proposed expression as $x \to \infty$ is 1, and the limit as $x \to -\infty$ is 0. To check right continuity of the proposed expression at $x = 0$ (obviously the only point to worry about), we note that $\lim_{\delta \searrow 0}[1 - C/(1+\delta^2)]1_{\{\delta \geq 0\}} = \lim_{\delta \searrow 0}[1 - C/(1+\delta^2)] = 1 - C = [1 - C/(1+0^2)]1_{\{0 \geq 0\}}$. The proposed expression is constant and hence nondecreasing on $(-\infty, 0)$. Since $1/(1+x)^2$ is nonincreasing on $[0,\infty)$, the proposed expression is nondecreasing on $[0,\infty)$ provided that $C \geq 0$. We also need $C \leq 1$ to ensure that the proposed expression does not decrease (and, indeed, become a negative number) as we pass over $x = 0$. In summary, the proposed expression is a valid cumulative distribution function if and only if $0 \leq C \leq 1$.

3. The analysis is similar to that in exercise 2 except for the check of right continuity at $x = 0$. Now we need $\lim_{\delta \searrow 0}[1 - C/(1+\delta^2)]1_{\{\delta \geq 0\}} = \lim_{\delta \searrow 0}[1 - C/(1+\delta^2)] = 1 - C$ to equal $0 = [1 - C/(1+0^2)]1_{\{0 > 0\}}$, which demands that $C = 1$.

4a. With $F_X(x) := [1 - 1/(1+x^2)]1_{\{x>0\}}$ the cumulative distribution function of X as determined in exercise 3, we have $f_X(x) := \frac{d}{dx}F_X(x) = 0$ for $x < 0$, $f_X(x) := \frac{d}{dx}F_X(x) = 2x/(1+x^2)^2$ for $x > 0$, and we can arbitrarily set $f_X(x)$ to 0 at $x = 0$. Then $f_X(x)$ is the probability density function of X.

4b. Put $y := g(x) := \sqrt{1+x^2}$ for $x > 0$, which is a strictly increasing function. We have $g^{-1}(y) = \sqrt{y^2 - 1}$ for $y > 1$ and hence $\frac{d}{dy}g^{-1}(y) = y/\sqrt{y^2-1}$ for $y > 1$. Then, for $y > 1$, we have $f_Y(y) = f_X(g^{-1}(y))\frac{d}{dy}g^{-1}(y) = 2y^{-3}$.

4c. Obviously $P(Z \leq z) = 0$ for $z < 0$. Otherwise, we have $P(Z \leq z) = P(|X-2| \leq z) = P(-z \leq X - 2 \leq z) = P(2 - z \leq X \leq 2 + z) = F_X(2+z) - F_X(2-z)$. If $0 \leq z \leq 2$, then $(2-z) \geq 0$ and we have $P(Z \leq z) = 1/(1+(2-z)^2) - 1/(1+(2+z)^2)$. If $z > 2$, then $(2-z) < 0$ and we have $P(Z \leq z) = 1 - 1/(1+(2+z)^2)$.

4d. For $z < 0$ we have $f_Z(z) := \frac{d}{dz}F_Z(z) = 0$. For $0 < z < 2$ we have $f_Z(z) := \frac{d}{dz}F_Z(z) = 2(2-z)/(1+(2-z)^2)^2 + 2(2+z)/(1+(2+z)^2)^2$. For $z > 2$ we have $f_Z(z) := \frac{d}{dz}F_Z(z) = 2(2+z)/(1+(2+z)^2)^2$. We can arbitrarily set $f_Z(z)$ to 0 at $z = 0$ and to $8/289 = 2(2+z)/(1+(2+z)^2)^2|_{z=2}$ at $z = 2$. Then $f_Z(z)$ is the probability density function of Z.