STA 623 — Fall 2010 — Dr. Charnigo

Solutions to Written Assignment 3

1a. Let W_3 denote the event that the third contestant wins, and let X denote the outcome of his/her first spin. We have $P(W_3|X=x)=1$ for $x \in \{90, 95, 100\}$ and $P(W_3|X=85)=0.5$ since a tiebreaker round is performed in the event that the second and third contestants have the same score. If the third contestant attains anything less than 85 on the first spin, then he/she will spin a second time. Three results on the second spin will yield a win (namely, those yielding a combined score from the two spins of 90, 95, or 100), one result will lead to a tiebreaker round (namely, that yielding a combined score from the two spins of 85), and the other sixteen results will yield a loss. Hence, $P(W_3|X=x)=0.175$ for $x \in \{5, 10, \ldots, 80\}$. Finally, since $P(X=x)=0.05$ for $x \in \{5, 10, \ldots, 100\}$, we conclude that

$$P(W_3)=\sum_{j=1}^{20} P(W_3|X=5j)P(X=5j)=0.315.$$

1b. If the second contestant scored 90, then $P(W_3|X=x)=1$ for $x \in \{95, 100\}$, $P(W_3|X=90)=0.5$, and $P(W_3|X=x)=0.125$ for $x \in \{5, 10, \ldots, 85\}$. As such,

$$P(W_3)=\sum_{j=1}^{20} P(W_3|X=5j)P(X=5j)=0.23125.$$

Similar computations yield $P(W_3)=0.1425$ if the second contestant scored 95 and $P(W_3)=0.04875$ if the second contestant scored 100.

1c. Let W_2 denote the event that you, the second contestant, win. Note that, since the first contestant has been disqualified, we have $W_2=W_3^c$. If you do not spin again, then 85 is your score and $P(W_2)=1-P(W_3)=0.685$ from part a. If you do spin again, then there are four possibilities: you spin a 5 for a total of 90 (call this event A_1), you spin a 10 for a total of 95 (call this event A_2), you spin a 15 for a total of 100 (call this event A_3), you spin more than 15 and are disqualified (call this event A_4). We have $P(A_1)=P(A_2)=P(A_3)=0.05$ and $P(A_4)=0.85$. Using part b to see that $P(W_2|A_1)=1-0.23125=0.76875$, $P(W_2|A_2)=1-0.1425=0.8575$, and $P(W_2|A_3)=1-0.04875=0.95125$, while noting that $P(W_2|A_4)$ is obviously 0, we obtain

$$P(W_2)=\sum_{j=1}^{4} P(W_2|A_j)P(A_j)=0.128875.$$

Since $0.128875 < 0.685$, we conclude that staying with 85 (not spinning again) is the better strategy.

Remark. Actually, we don't even need to use part b to answer part c. If you do spin again, you can't win with event A_4, so $W_2 \subset A_4^c$ and $P(W_2) \leq P(A_4^c)=1-0.85=0.15<0.685$.

2a. For $w>0$ we have

$$P(W \leq w)=P(\max\{T,U\} \leq w)=P(T \leq w \;\cap\; U \leq w)=P(T \leq w)P(U \leq w)$$

$$=(1-\exp[-\lambda w])(1-\exp[-\mu w])=1-\exp[-\lambda w]-\exp[-\mu w]+\exp[-(\lambda+\mu)w].$$

(Obviously $P(W \leq w)=0$ for $w \leq 0$.) We also record here for use in part c that the probability density function of W, obtained by differentiation of the cumulative distribution function, is

$$f_W(w):=(\lambda\exp[-\lambda w]+\mu\exp[-\mu w]-(\lambda+\mu)\exp[-(\lambda+\mu)w])1_{\{w>0\}}.$$

2b. For $w>0$ we have

$$P(W>w)=\exp[-\lambda w]+\exp[-\mu w]-\exp[-(\lambda+\mu)w].$$

Since

$$\lim_{M\to\infty} MP(W > M) = \lim_{M\to\infty} (M\exp[-\lambda M] + M\exp[-\mu M] - M\exp[-(\lambda+\mu)M]) = 0$$

(L'Hopital's rule shows that, in fact, $\lim_{M\to\infty} M^a \exp[-bM] = 0$ for any positive real numbers a and b), we may use the alternative expectation formula from Section 2.2 to obtain

$$E[W] = \int_0^\infty (\exp[-\lambda w] + \exp[-\mu w] - \exp[-(\lambda+\mu)w])\, dw = 1/\lambda + 1/\mu - 1/(\lambda+\mu).$$

2c. Put $Y := \exp[-W]$. Since $\mathcal{W} = (0,\infty)$, we have $\mathcal{Y} = (0,1)$. Then, with $y = g(w) := \exp[-w]$ and $w = g^{-1}(y) := -\log y$ for $w \in (0,\infty)$ and $y \in (0,1)$, we obtain the probability density function of Y as

$$f_Y(y) = f_W(g^{-1}(y)) \left|\frac{d}{dy} g^{-1}(y)\right| = (\lambda y^\lambda + \mu y^\mu - (\lambda+\mu) y^{\lambda+\mu}) \left|-\frac{1}{y}\right| = \lambda y^{\lambda-1} + \mu y^{\mu-1} - (\lambda+\mu) y^{\lambda+\mu-1}$$

for $y \in (0,1)$.

Remark. An alternative solution entails noting that, for $y \in (0,1)$,

$$P(Y \le y) = P(\exp[-W] \le y) = P(W \ge -\log y) = 1 - P(W \le -\log y) = y^\lambda + y^\mu - y^{\lambda+\mu}.$$

This expression can then be differentiated to obtain $f_Y(y)$.

3a. Once $n > e\lambda$ we have $p_n = 1 - \lambda/n$ and $1 - p_n = \lambda/n < 1/e$. Then

$$M_{X_n}(t) = \left(\frac{p_n}{1 - (1-p_n)\exp[t]}\right)^{r_n} = \left(\frac{1-\lambda/n}{1-\exp[t]\lambda/n}\right)^n$$

for all $t < -\log[1-p_n]$. Since $1 - p_n < 1/e$ we have $-\log[1-p_n] > -\log[1/e] = 1$, meaning that the above expression for $M_{X_n}(t)$ is valid for all $t \in [-h, h]$ with $h := 1$. Writing $1-\lambda/n = 1-\exp[t]\lambda/n+\lambda/n(\exp[t]-1)$, we have

$$M_{X_n}(t) = \left(\frac{1-\exp[t] + \lambda/n(\exp[t]-1)}{1-\exp[t]\lambda/n}\right)^n = \left(1 + \frac{\lambda(\exp[t]-1)/(1-\exp[t]\lambda/n)}{n}\right)^n,$$

whose limit as $n \to \infty$ is

$$\exp\left[\lim_{n\to\infty} \lambda(\exp[t]-1)/(1-\exp[t]\lambda/n)\right] = \exp\left[\lambda(\exp[t]-1)\right] = M_X(t)$$

for X a Poisson random variable with mean λ.

3b. From the third result on moment generating functions, we have

$$\lim_{n\to\infty} P(X_n \le x) = P(X \le x)$$

for all x at which $P(X \le x)$ is continuous. Since X is a discrete random variable with support set the nonnegative integers, the limit is valid for all x except nonnegative integers. With $x < 0$ the limit is, of course, 0. With $x > 0$ the limit equals $P(X \le \lfloor x \rfloor) = \sum_{j=0}^{\lfloor x \rfloor} \exp[-\lambda]\lambda^j/j!$.

Remark. Actually, in this particular problem, the limit works even for x at which $P(X \le x)$ is discontinuous. To see that, let x be a nonnegative integer. Then, since $x + 0.5$ is not a nonnegative integer, we have

$$\lim_{n\to\infty} P(X_n \le x + 0.5) = P(X \le x + 0.5).$$

Yet, $P(X_n \le x + 0.5) = P(X_n \le x)$ for all $n > e\lambda$ and $P(X \le x + 0.5) = P(X \le x)$, so indeed we have

$$\lim_{n\to\infty} P(X_n \le x) = P(X \le x).$$

For an example in which the limit does not work for x at which $P(X \le x)$ is discontinuous, take X_n to be normally distributed with mean 0, variance $1/n$ and X to equal 0 with probability 1. Then $\lim_{n\to\infty} M_{X_n}(t) = 1 = M_X(t)$ for all $t \in \mathbb{R}$ and $P(X_n \le 0) = 1/2$ for all n but $P(X \le 0) = 1 \ne 1/2 = \lim_{n\to\infty} P(X_n \le 0)$.