# STA 623 — Fall 2010 — Dr. Charnigo

## Solutions to Written Assignment 4

1a. Since $p \in (0,1)$, we have $\log p < 0$ and $-1/\log p > 0$, from which nonnegativity of $f(x;p)$ follows. Therefore, to show that $f(x;p)$ defines a valid probability mass function, all we need to do is verify that the summation over its support set is 1. We have

$$\begin{aligned}
\sum_{x=1}^{\infty} -(1-p)^x/(x \log p) &= (-1/\log p) \sum_{x=1}^{\infty} (1-p)^x/x \\
&= (-1/\log p) \sum_{x=1}^{\infty} (-q)^x/x \\
&= (-1/\log p)(-\log[1+q]) \\
&= (-1/\log p)(-\log p) \\
&= 1
\end{aligned}$$

with $q := p - 1 \in (-1, 0) \subset (-1, 1)$ for which the series expansion invoked in the third equality is valid.

1b. We have

$$E[X] = \sum_{x=1}^{\infty} x \ f(x;p) = (-1/\log p) \sum_{x=1}^{\infty} (-q)^x = (-1/\log p)(-q/(1+q)) = (p-1)/(p \log p)$$

with $q := p - 1 \in (-1, 0) \subset (-1, 1)$ for which the series expansion invoked in the third equality is valid.

1c. We have

$$\begin{aligned}
E[X^2] &= \sum_{x=1}^{\infty} x^2 \ f(x;p) \\
&= (-1/\log p) \sum_{x=1}^{\infty} x q^x \\
&= (-1/\log p) \sum_{x=1}^{\infty} x q^{x-1} q \\
&= (-q/\log p) \sum_{x=0}^{\infty} x q^{x-1} \\
&= (-q/\log p)/(1-q)^2 \\
&= (p-1)/(p^2 \log p),
\end{aligned}$$

with $q := 1 - p \in (0, 1)$ for which the series expansion invoked in the fifth equality is valid. As such,

$$Var[X] = E[X^2] - (E[X])^2 = \left( \frac{p-1}{p^2 \log p} \right) \left( 1 - \frac{p-1}{\log p} \right).$$

1d. We verify that $f(x;p)$ belongs to an exponential family by writing $f(x;p) = h(x)c(p)\exp[w(p)t(x)]$ with $h(x) := 1_{\{x \in \{1,2,\ldots\}\}}/x$, $c(p) := -1/\log p$, $w(p) := \log(1-p)$, and $t(x) := x$.

1e. We have

$$-\frac{d}{dp} \log c(p) = -\frac{d}{dp} \log \left( \frac{-1}{\log p} \right) = \frac{d}{dp} \log(-\log p) = \frac{-1/p}{-\log p} = \frac{1}{p \log p},$$

$$-\frac{d^2}{dp^2}\log c(p) = \frac{d}{dp}\left(\frac{1}{p\log p}\right) = \frac{d}{dp}(p\log p)^{-1} = -(p\log p)^{-2}(1+\log p) = \frac{-(1+\log p)}{p^2(\log p)^2}.$$

Moreover,

$$\frac{d}{dp}w(p) = \frac{d}{dp}\log(1-p) = \frac{-1}{1-p}$$

and

$$\frac{d^2}{dp^2}w(p) = \frac{d}{dp}\left(\frac{-1}{1-p}\right) = \frac{-1}{(1-p)^2}.$$

As such, the equation

$$E\left[\frac{d}{dp}w(p)\ t(X)\right] = -\frac{d}{dp}\log c(p)$$

yields

$$\left(\frac{-1}{1-p}\right)E[X] = \frac{1}{p\log p},$$

from which we obtain

$$E[X] = \frac{(p-1)}{p\log p}.$$

Further, the equation

$$Var\left[\frac{d}{dp}w(p)\ t(X)\right] = -\frac{d^2}{dp^2}\log c(p) - E\left[\frac{d^2}{dp^2}w(p)\ t(X)\right]$$

yields

$$\left(\frac{-1}{1-p}\right)^2 Var[X] = \frac{-(1+\log p)}{p^2(\log p)^2} + \frac{1}{(1-p)^2}E[X] = \frac{-(1+\log p)}{p^2(\log p)^2} + \frac{-1}{(1-p)p\log p},$$

from which we obtain

$$Var[X] = \left(\frac{p-1}{p^2\log p}\right)\left(1-\frac{p-1}{\log p}\right).$$

2. Put $f(x) := 1_{\{0<x<1\}}$, $\mathcal{F} := \{1_{\{a<x<b\}}/(b-a) : -\infty < a < b < \infty\}$, and $\mathcal{G} := \{\sigma^{-1}f((x-\mu)\sigma^{-1}) : \mu \in (-\infty,\infty), \sigma \in (0,\infty)\}$. Since $\mathcal{G}$ is obviously a location-scale family, we will show that $\mathcal{F}$ is a location-scale family by showing that $\mathcal{F} = \mathcal{G}$.

Pick a member of $\mathcal{F}$, $1_{\{a<x<b\}}/(b-a)$, with $-\infty < a < b < \infty$. By setting $\mu := a$ and $\sigma := b-a > 0$, we find that

$$1_{\{a<x<b\}}/(b-a) = 1_{\{0<x-a<b-a\}}/(b-a) = 1_{\{0<(x-a)/(b-a)<1\}}/(b-a) = \sigma^{-1}f((x-\mu)\sigma^{-1}),$$

which is to say that $1_{\{a<x<b\}}/(b-a)$ is also a member of $\mathcal{G}$. Hence, $\mathcal{F} \subset \mathcal{G}$.

Now pick a member of $\mathcal{G}$, $\sigma^{-1}f((x-\mu)\sigma^{-1})$, with $\mu \in (-\infty,\infty)$ and $\sigma \in (0,\infty)$. By setting $a := \mu$ and $b := \mu+\sigma > a$, we find that

$$\sigma^{-1}f((x-\mu)\sigma^{-1}) = \sigma^{-1}1_{\{0<(x-\mu)\sigma^{-1}<1\}} = \sigma^{-1}1_{\{0<x-\mu<\sigma\}} = \sigma^{-1}1_{\{\mu<x<\mu+\sigma\}} = 1_{\{a<x<b\}}/(b-a),$$

which is to say that $\sigma^{-1}f((x-\mu)\sigma^{-1})$ is also a member of $\mathcal{F}$. Hence, $\mathcal{G} \subset \mathcal{F}$.

3. Chebychev's Inequality yields, for any $\delta \in (0,\infty)$,

$$P(|X| \geq \delta) = P(|X|^{1/3} \geq \delta^{1/3}) \leq \delta^{-1/3}E[\ |X|^{1/3}\ ].$$

Hence,

$$P(|X| \geq \delta) \leq \delta^{-1/3}\int_{\mathbb{R}}\frac{|x|^{1/3}}{\pi(1+x^2)}\ dx = \frac{2}{\pi}\delta^{-1/3}\int_0^\infty \frac{x^{1/3}}{1+x^2}\ dx = \frac{2}{\pi}\delta^{-1/3}\frac{\pi}{\sqrt{3}} = \frac{2}{\sqrt{3}}\delta^{-1/3}.$$