

# STA 623 – Fall 2011 – Dr. Charnigo

## Section 1.6: Density and Mass Functions

*Probability mass function.* Let  $X$  be a discrete random variable. We define the probability mass function of  $X$  as  $f(x) := P(X = x)$  for any  $x \in \mathbb{R}$ . Note that (lower case)  $x$  is a placeholder for 2 or 8 or some other number, so that (for example)  $f(2) = P(X = 2)$  and  $f(8) = P(X = 8)$ .

A probability mass function must satisfy  $f(x) \geq 0$  for all  $x \in \mathbb{R}$  and  $\sum_{x \in \mathbb{R}: f(x) > 0} f(x) = 1$ . Conversely, any function  $f(x)$  with these properties may be interpreted as a probability mass function.

For a discrete random variable  $X$  with probability mass function  $f(x)$  and cumulative distribution function  $F(x)$ , we have  $P(X \in B) = \sum_{x \in B: f(x) > 0} f(x)$  for any set  $B \in \mathcal{B}^1$ , the sigma field on  $\mathbb{R}$  generated by its open subintervals. In particular,

$$\begin{aligned} P(a \leq X \leq b) &= \sum_{x \in [a, b]: f(x) > 0} f(x) \\ &\geq P(a < X \leq b) = \sum_{x \in (a, b]: f(x) > 0} f(x) = F(b) - F(a) \\ &\geq P(a < X < b) = \sum_{x \in (a, b): f(x) > 0} f(x) \end{aligned}$$

for any  $a, b \in \mathbb{R}$  with  $a < b$ . The first “ $\geq$ ” above is “ $>$ ” if  $f(a) > 0$  and “ $=$ ” otherwise. The second “ $\geq$ ” above is “ $>$ ” if  $f(b) > 0$  and “ $=$ ” otherwise.

**Example (probability mass function).** Let  $\lambda$  be a positive number and put  $f(x) := 1_{\{x \in \{0, 1, 2, \dots\}\}} C(\lambda) \lambda^x / x!$ . How can  $C(\lambda)$  be chosen so that  $f(x)$  is a probability mass function?

Let  $X$  be a random variable with this probability mass function, which we may write in shorthand as  $X \sim f(x)$ . What is  $P(X > 1)$ ? What is  $P(X \text{ even})$ ?

*Probability density function.* Let  $X$  be a continuous random variable with cumulative distribution function  $F(x)$ . Suppose that there exists a function  $f(x)$  such that  $F(x) = \int_{-\infty}^x f(t) dt$ . Then we refer to  $f(x)$  as a probability density function of  $X$ . Since altering  $f(x)$  at finitely many points has no impact on its integration, a probability density function is not unique.

A probability density function must satisfy  $\int_{-\infty}^{\infty} f(t) dt = 1$  and cannot be negative over any interval of nonzero length. We may as well assume, as is routinely done, that  $f(x) \geq 0$  for all  $x \in \mathbb{R}$ . Conversely, any function  $f(x)$  with these properties may be interpreted as a probability density function.

If  $f(x)$  is continuous, then  $f(x) = \frac{d}{dx}F(x)$ .

For a continuous random variable  $X$  with probability density function  $f(x)$  and cumulative distribution function  $F(x)$ , we have

$$P(a < X < b) = \int_a^b f(x) dx$$

for any  $a, b \in \mathbb{R}$  with  $a < b$ . The above equality also holds with  $a = -\infty$  and/or  $b = \infty$ . For finite  $a$  and  $b$ , we have the additional equalities

$$P(a < X < b) = P(a \leq X < b) = P(a < X \leq b) = P(a \leq X \leq b) = F(b) - F(a).$$

**Example (probability density function).** Let  $\lambda$  be a positive number and put  $f(x) := 1_{\{x>0\}}C(\lambda) \exp[-\lambda x]$ . How can  $C(\lambda)$  be chosen so that  $f(x)$  is a probability density function?

Let  $X$  be a random variable with this probability density function, which we may write in shorthand as  $X \sim f(x)$ . What is  $P(X > 1)$ ? What is  $F(x)$ ? Note that  $f(x)$  is discontinuous only at 0 and that  $f(x) = \frac{d}{dx}F(x)$  for all  $x \neq 0$ .

*Additional practice.* Let  $X$  be a random variable with probability density function  $f(x) := 1_{\{x>0\}}(\log 2) \exp[-(\log 2)x]$ . Show that, for any positive reals  $a, b$ , we have

$$P(X \geq a + b \mid X \geq a) = P(X \geq b \mid X \geq 0).$$

If  $X$  is used to model the lifetime of a light bulb in months, then the preceding equality says that a light bulb that has been in use for  $a$  months is just as likely to last another  $b$  months as is a fresh light bulb.

Let  $Y := \lceil X \rceil$ , read “the ceiling of  $X$ ”, which means the least integer greater than or equal to  $X$ . What is the probability mass function of  $Y$ ? Show that, for any positive integers  $a, b$ , we have

$$P(Y > a + b \mid Y > a) = P(Y > b \mid Y > 0).$$

If  $Y$  is used to model the number of coin flips required to obtain one’s first “Heads”, then the preceding equality says that a person who has already started flipping a coin and obtained “Tails” on each of the first  $a$  attempts is just as likely to get all “Tails” in the next  $b$  attempts as is another person who has just started flipping a coin.