STA 623 — Fall 2011 — Dr. Charnigo

Written Assignment 1

Written Assignment 1 is due on Tuesday 20 September at the end of class. You are encouraged to work in groups of two or three, but you may work individually if you prefer. In what follows, all sets are assumed to be events that belong to the sigma field.

[10] 1. Give a careful proof of the second distributive law, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

[10] 2. Prove result 9 that $P(A) \leq P(B)$ whenever $A \subset B$, without using any subsequent result. (Thus, you are limited to axioms 1, 2, 3 and results 4, 5, 6, 7, 8.)

[10] 3. I will be dealt 7 cards from a well-shuffled standard 52-card deck. What is the probability that I will receive one triple and two pairs?

[10] 4. Consider two lottery games. Game A entails selecting 3 balls without replacement from a vat containing 30 balls numbered from 1 to 30; you win if you guess at least two of the numbers correctly. Game B entails selecting 6 balls without replacement from a vat containing 60 balls numbered 1 to 60; you win if you guess at least four of the numbers correctly. With which game do you have a better chance of winning?

[10] 5. Provide an example in which events A, B, C are not independent even though A, B are independent, A, C are independent, and B, C are independent.

[10] 6. Sometimes people use a Bonferroni correction in post-hoc testing after one-way analysis of variance. This entails setting the significance level of each post-hoc test to α/m, where m is the total number of post-hoc tests to be performed. The idea is that the probability of incorrectly rejecting at least one true null hypothesis will then be no more than α.

For $i \in \{1, 2, \ldots, m\}$, let A_i denote the event that the null hypothesis in post-hoc test i is rejected and B_i denote the event that the null hypothesis in post-hoc test i is true. Justify the Bonferroni correction by proving that $P(\cup_{i=1}^{m} \{A_i \cap B_i\}) \leq \alpha$.

[20] 7. Suppose that 0.8% (8/10 of 1%, not 80%) of people in the U.S. have HIV and that a diagnostic test is available such that: (i) $P\%$ of people who really have HIV test positive; and, (ii) $Q\%$ of people who really do not have HIV test negative, where P and Q are some numbers between 0 and 100. Indicate whether each of the following statements is true or false; justify your answers.

a. If $P = 100$, then there are no false positive test results.
b. If $P = 100$, then there are no false negative test results.
c. If $Q = 100$, then there are no false positive test results.
d. If $Q = 100$, then there are no false negative test results.

[20] 8. In each of the following cases, for what value(s) of C (if any) is $F(x)$ a valid cumulative distribution function for a continuous random variable? a discrete random variable? a random variable that is neither discrete nor continuous?

a. $F(x) := \sin(Cx)1_{x\in[0,1]} + 1_{x>1}$
b. $F(x) := \cos(Cx)1_{x\in[0,1]} + 1_{x>1}$
c. $F(x) := \sin(Cx)1_{x\in(0,1)} + 1_{x\geq 1}$
d. $F(x) := \cos(Cx)1_{x\in(0,1)} + 1_{x\geq 1}$

Remark. I now prove one of the claims from p. 3 of the script for Section 1.1.

Suppose that $x \in \cap_{n=1}^{\infty} \cup_{i=n}^{\infty} A_i$. Then $x \in \cup_{i=1}^{\infty} A_i$, so there exists a positive integer m_1 such that $x \in A_{m_1}$. Since $x \in \cup_{i=m_1+1}^{\infty} A_i$, there also exists a positive integer $m_2 > m_1$ such that $x \in A_{m_2}$. Since $x \in \cup_{i=m_2+1}^{\infty} A_i$, there also exists a positive integer $m_3 > m_2$ such that $x \in A_{m_3}$. This argument can be repeated indefinitely, leading us to conclude that there exists a strictly ascending sequence of positive integers $\{m_j\}_{j=1}^{\infty}$ such that $x \in A_{m_j}$ for every positive integer j. In other words, x belongs to infinitely many of $A_1, A_2, A_3, \ldots$.

Now suppose that there exists a strictly ascending sequence of positive integers $\{m_j\}_{j=1}^{\infty}$ such that $x \in A_{m_j}$ for every positive integer j. Let n be a positive integer. Then there exists a number in the sequence $\{m_j\}_{j=1}^{\infty}$ that is greater than n, call it $m_{j(n)}$. We have $x \in A_{m_{j(n)}}$ and therefore $x \in \cup_{i=n}^{\infty} A_i$. Since the preceding statement holds for every positive integer n, we have $x \in \cap_{n=1}^{\infty} \cup_{i=n}^{\infty} A_i$.